TMA195 ALA-B 2000, studio 3.1, obligatory work[1]

1 Averages

Key words: integral, average, mean value theorem, weight, change of variable, center of mass, remainder in Taylor's formula.

1.1 The average of a function

(See EM2000 Ch 28.11, 28.14) Let the function f be Lipschitz continuous (or piecewise Lipschitz continuous) on the interval $[a, b]$, so that the integral $\int_a^b f(x)\, dx$ is defined. The number

$$\bar{f} = \frac{1}{b-a} \int_a^b f(x)\, dx \tag{1}$$

is called the *average* or *mean value* (medelvärde) of the function f over the interval $[a, b]$.

The Mean Value Theorem for integrals (EM2000 Ch 28.14) says that, if f is Lipschitz continuous on $[a, b]$ (piecewise Lipschitz continuous is not sufficient here!), then there is a point $\bar{x} \in [a, b]$, where f is equal to its average, $f(\bar{x}) = \bar{f}$, i.e.,

$$f(\bar{x}) = \frac{1}{b-a} \int_a^b f(x)\, dx. \tag{2}$$

We repeat the proof (see EM2000 Ch 28.11).

Proof. Assume that

$$f(x) > \bar{f}, \quad \forall x \in [a, b]. \tag{3}$$

Then

$$\bar{f} = \frac{1}{b-a} \int_a^b f(x)\, dx > \frac{1}{b-a} \int_a^b \bar{f}\, dx = \bar{f}. \tag{4}$$

This is a contradiction and (3) must be false. Therefore there is a point $c \in [a, b]$ such that $f(c) \leq \bar{f}$. In the same way we prove that there is a point $d \in [a, b]$ such that $f(d) \geq \bar{f}$. Thus, the value $\bar{f}$ is intermediate between $f(c)$ and $f(d)$:

$$f(c) \leq \bar{f} \leq f(d). \tag{5}$$

Since f is Lipschitz continuous on $[a, b]$ it follows from the Intermediate Value Theorem that there is $\bar{x} \in [c, d]$, where f takes the value $\bar{f}$, i.e., $f(\bar{x}) = \bar{f}$. □

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Example 1. We compute the average of the function $f(x) = x^2$ on the interval $[-2, 2]$:

$$\bar{f} = \frac{1}{4}\int_{-2}^{2} x^2\,dx = \frac{1}{4}\Big[\frac{x^3}{3}\Big]_{-2}^{2} = \frac{4}{3}. \tag{6}$$

We note that there are two points in $[-2, 2]$, where $x^2 = 4/3$.

Problem 1. Compute the average of the function f over the interval $[a, b]$. Draw the graph of f and the average. At how many points is f equal to its average?

$$\begin{aligned}
&(a) \quad f(x) = x^3 - x, \quad [a,b] = [-2,2]\\
&(b) \quad f(x) = x^3 - x, \quad [a,b] = [0,2]\\
&(c) \quad f(x) = \begin{cases} -1, & x < 0 \\ x, & x > 0 \end{cases}, \quad [a,b] = [-1,1]
\end{aligned}$$

(I don't give the answers, because it is a very important part of such an exercise to convince yourself that you have done it right. So you must do each problem several times, in different ways, by hand computation, by matlab computation, by plotting the functions by hand and using matlab, `fplot('funk',[a,b],'r')`, until you have squeezed out every bit of training that you can from the exercise. The answers are not important here, what's important is the various kinds of training that you can get by working out the problems and by writing your work down. I think it is better to do one problem three times, reflecting about it, than to do three different problems once. Remember that you set `ua=0` in `my_int`, if you only want the integral, so that $u(b) = u_a + \int_a^b f(x)\,dx = \int_a^b f(x)\,dx$.)

1.2 Weighted average

Example 2. A math professor is assigning the final grades for the course Analysis and Linear Algebra, which consists of three parts. Parts A and C are worth $w_A = w_C = 4$ credit points, part B is worth $w_B = 8$ credit points. If the student obtains the grades f_A, f_B, f_C, then the final grade is the *weighted average*

$$\bar{f} = \frac{f_A w_A + f_B w_B + f_C w_C}{w_A + w_B + w_C} = \frac{\sum\limits_{j=A,B,C} f_j w_j}{\sum\limits_{j=A,B,C} w_j}. \tag{7}$$

Suppose that a student scores $f_A = 3$, $f_B = 4$, $f_C = 3$. Then

$$\bar{f} = \frac{3\cdot 4 + 4\cdot 8 + 3\cdot 4}{4+8+4} = 3.5. \tag{8}$$

This can be formulated with integrals if we define $[a,b]=[0,3]$ and

$$w(x)=\begin{cases} w_A=4, & x\in(0,1),\\ w_B=8, & x\in(1,2),\\ w_C=4, & x\in(2,3),\end{cases}\qquad f(x)=\begin{cases} f_A, & x\in(0,1),\\ f_B, & x\in(1,2),\\ f_C, & x\in(2,3).\end{cases} \tag{9}$$

(Note that w and f are piecewise constant functions, and hence piecewise Lipschitz.) Then

$$\bar{f}=\frac{\int_a^b f(x)w(x)\,dx}{\int_a^b w(x)\,dx}. \tag{10}$$

Suppose, again, that a student scores $f_A=3$, $f_B=4$, $f_C=3$. Then

$$\bar{f}=\frac{\int_0^1 3\cdot 4\,dx+\int_1^2 4\cdot 8\,dx+\int_2^3 3\cdot 4\,dx}{\int_0^1 4\,dx+\int_1^2 8\,dx+\int_2^3 4\,dx}=3.5, \tag{11}$$

which is (probably) rounded to 4. This shows how important it is to do well on Part B. Note that there is no point $\bar{x}$, where f is equal to its average. Why?

A function w is called a *weight function* (viktfunktion) on $[a,b]$ if it satisfies

$$\begin{aligned} &w \text{ is piecewise Lipschitz continuous on } [a,b],\\ &w(x)>0,\quad \forall x\in[a,b]. \end{aligned} \tag{12}$$

The total weight is then $W_{[a,b]}=\int_a^b w(x)\,dx$. Assume that f is piecewise Lipschitz continuous on $[a,b]$. The average of f over $[a,b]$ with respect to the weight w is defined as

$$\bar{f}=\frac{1}{W_{[a,b]}}\int_a^b f(x)w(x)\,dx. \tag{13}$$

There is a Mean Value Theorem also for weighted averages.

Theorem 1 (Mean Value Theorem for Integrals with Weight). If f is Lipschitz continuous on $[a,b]$ and w is a weight function, then there is (at least) one point $\bar{x}\in[a,b]$ such that

$$f(\bar{x})=\frac{1}{W_{[a,b]}}\int_a^b f(x)w(x)\,dx,\quad \text{where } W_{[a,b]}=\int_a^b w(x)\,dx. \tag{14}$$

Proof. Let $W(x)=\int_a^x w(y)\,dy$ be a primitive function of w. We make a change of variable in (13) (see EM2000 Ch 28.8):

$$z=W(x),\ dz=W'(x)\,dx=w(x)\,dx,\ W(a)=0,\ W(b)=W_{[a,b]}. \tag{15}$$

Since $w(x) > 0$ for all x, we know that W strictly increasing (EM2000 Ch 28.12), so that it has an inverse function $x = W^{-1}(z)$. We get

$$\bar{f} = \frac{1}{W_{[a,b]}} \int_a^b f(x)w(x)\,dx = \frac{1}{W_{[a,b]}} \int_0^{W_{[a,b]}} f(W^{-1}(z))\,dz. \tag{16}$$

The mean value theorem, see (2), gives a point $\bar{z} \in [0, W_{[a,b]}]$ such that

$$\frac{1}{W_{[a,b]}} \int_0^{W_{[a,b]}} f(W^{-1}(z))\,dz = f(W^{-1}(\bar{z})) = f(\bar{x}). \tag{17}$$

□

A weight function does not have to be strictly positive as in (12). In fact, (12) can be replaced by

$$\begin{aligned} &w \text{ is piecewise Lipschitz continuous on } [a,b], \\ &w(x) \geq 0, \quad \forall x \in [a,b], \quad \int_a^b w(x)\,dx > 0. \end{aligned} \tag{18}$$

See Problem 5.

Problem 2. Compute the average of the function f over the interval $[a, b]$ with respect to the weight w. Draw the graph of f, w, and the average. At how many points is f equal to its average?

$$\begin{aligned} &(a) \quad f(x) = x^3 - x, \quad [a,b] = [0,2], \quad w(x) = 1 \\ &(b) \quad f(x) = x^3 - x, \quad [a,b] = [0,2], \quad w(x) = 2 - x \\ &(c) \quad f(x) = x, \quad [a,b] = [0,3], \quad w \text{ as in (9)} \end{aligned}$$

Problem 3. Perform the steps of the proof of Theorem 1 with the weight function in Problem 2 (b).

Problem 4. Compute the primitive function $W(x) = \int_0^x w(y)\,dy$ of w in (9). Draw its graph and use this to check that the inverse W^{-1} exists. Perform the steps of the proof of Theorem 1 with this weight function.

Problem 5. (advanced, not obligatory) Modify the proof of Theorem 1 to the following situation. Suppose that w satisfies (18) and that $w(x) = 0$ for all $x \in [c, d]$, where $a < c < d < b$, and $w(x) > 0$ elsewhere. Hint: $W(c) = W(d)$.

1.3 Center of mass

(Not obligatory.) Consider a rod of length L, constant width b, and variable height $h = h(x)$ for $x \in [0, L]$. These have the unit of length: `[m]`. The density of mass is constant ρ `[kg/m³]`. The total mass is then given by

$$M = \int_0^L \rho b h(x)\,dx = \rho b \int_0^L h(x)\,dx. \quad \texttt{[kg]} \tag{19}$$

It is useful to think of the quantity $dV = bh(x)\,dx$ `[m³]` as a *volume element* of infinitesimal thickness dx, so that $M = \int \rho\, dV$. Draw a figure that illustrates this.

The mean value theorem, with weight $w(x) = \rho b h(x)$, says that there is $\bar{x} \in [0, L]$ such that

$$\bar{x} = \frac{1}{M}\int_0^L x\rho b h(x)\,dx = \frac{\rho b \displaystyle\int_0^L x h(x)\,dx}{\rho b \displaystyle\int_0^L h(x)\,dx} = \frac{\displaystyle\int_0^L x h(x)\,dx}{\displaystyle\int_0^L h(x)\,dx}. \tag{20}$$

The point $\bar{x}$ is called the *center of mass* (masscentrum, tyngdpunkt) of the rod.

We may also consider a rod with constant height but variable density $\rho(x)$. Then

$$\bar{x} = \frac{1}{M}\int_0^L x\rho(x) b h\,dx, \quad \text{where } M = \int_0^L \rho(x) b h\,dx. \tag{21}$$

Problem 6. Compute the center of mass of a tapered rod: length L, width b, and height $h(x) = H(\alpha + \beta x/L)$, where H, α, β are constants, α, β are dimensionless numbers. What is the dimension of H? Important training: compute the integrals both with and without the change of variable $z = x/L$. Check your result against the special cases $\alpha = 1$, $\beta = 0$ and $\alpha = 0$, $\beta = 1$, which are easier.

1.4 The remainder in Taylor's formula

(See EM2000 Ch 28.9, Problem 28.11) The remainder in Taylor's formula is

$$R_n(x, \bar{x}) = \int_{\bar{x}}^{x} \frac{(x-y)^n}{n!} u^{(n+1)}(y)\,dy. \tag{22}$$

Use the mean value theorem (Theorem 1) to show that there is a point $\hat{x}$ between $\bar{x}$ and x such that

$$R_n(x, \bar{x}) = \frac{(x-\bar{x})^{n+1}}{(n+1)!} u^{(n+1)}(\hat{x}). \tag{23}$$

Hint: consider first the case when $x > \hat{x}$, use the weight $w(y) = (x-y)/n!$, and find $\hat{x} \in [\bar{x}, x]$. What is the primitive function W now? Then consider the case when $x < \bar{x}$.

Note that if $|u^{(n+1)}(x)| \le M$ for all $x \in [a, b]$, then

$$|R_n(x, \bar{x})| = M\frac{|x-\bar{x}|^{n+1}}{(n+1)!}, \tag{24}$$

so that the remainder term is smaller than the other terms in Taylor's formula when x is close to $\bar{x}$.

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