86
Incompressible Navier-Stokes: quick and easy

A computable solution certainly exists. I don't care about the possible existence of uncomputable solutions. (Jack Nicolson)

Thus the methods of Lagrange and Hamilton are undoubtedly *useful* in helping us to carry out the primary task of dynamics - namely, to find out how systems move. But it would be wrong to think that this is the sole purpose of these general methods or even their main purpose. They do much more. In fact, they teach us what dynamics *really is* : It is the study of certain types of differential equations. (Synge and Griffiths, Principles of Mechanics, 1959)

My attention (was) drawn to various mechanical phenomena, for the explanation of which I discovered that a knowledge of mathematics was essential. (Reynolds)

By this research it is shown that there is one, and only one, conceivable purely mechanical system capable of accounting for all the physical evidence, as we know it in the Universe. (Reynolds)

86.1 Introduction

The Navier-Stokes equations is the basic model for fluid flow and describe a variety of phenomena in hydro and aero-dynamics, processing industry, biology, oceanography, geophysics, meteorology and astrophysics. Fluid flow in all these applicatons usually contains features of both *turbulent* and *laminar* flow, with turbulent flow being irregular with rapid fluctuations in space and time and laminar flow being more organized. The basic question of *Computational Fluid Dynamics* CFD is how to efficiently and reliably solve the Navier-Stokes equations numerically for both laminar and turbulent flow.

The Navier-Stokes equations is a system of nonlinear differential equations coupling the phenomena of convection and diffusion. Traditionally, the study of the Navier-Stokes equations is separated into *incompressible* and *compressible* flow, using different dependent variables: *primitive variables* (velocity, pressure, temperature) for incompressible flow and *conservation variables* (density, momentum, energy) for compressible flow. We focus in this chapter on the incompressible Navier-Stokes equations in the case of constant density, viscosity and temperature, with the velocity and pressure as variables. We present the cG(1)dG(0) finite element method with cG(1) in space and dG(0) in time, and follow up with the corresponding cG(1)dG(1) and cG(1)cG(1) methods.

86.2 The incompressible Navier-Stokes equations

The Navier-Stokes equations for an incompressible Newtonian fluid with constant kinematic viscosity $\nu > 0$, unit density and constant temperature enclosed in a volume Ω in $\mathbb{R}^3$ with boundary Γ, take the form: find the velocity (u, p) such that

$$\begin{array}{rcll} \frac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu\Delta u + \nabla p &=& f & \text{in } \Omega \times I, \\ \nabla \cdot u &=& 0 & \text{in } \Omega \times I, \\ u &=& w & \text{on } \Gamma \times I, \\ u(\cdot, 0) &=& u^0 & \text{in } \Omega, \end{array} \qquad (86.1)$$

where $u = (u_1, u_2, u_3)$ is the velocity and p the pressure of the fluid and f, w, u^0, $I = (0, T)$, is a given driving force, boundary data, initial data and time interval, respectively. Recall that

$$\frac{\partial v}{\partial t} + (u \cdot \nabla)v = \frac{\partial v}{\partial t} + \sum_{i=1}^{3} u_i \frac{\partial v}{\partial x_i} \qquad (86.2)$$

is the *particle derivative* of a quantity $v(x, t)$ measuring the rate of change of $v(x(t), t)$ with respect to time, that is the rate of change of v along a trajectory $x(t)$ of a fluid particle with velocity $u(x, t)$, satisfying $\frac{dx}{dt} = u(x(t), t)$. In particular, $\frac{\partial u}{\partial t} + (u \cdot \nabla)u$ is the acceleration (rate of change of velocity) of a fluid particle. The expression $\nu\Delta u - \nabla p$ represents the total force on a fluid particle resulting from of viscous shear force and an isotropic pressure. The first equation of 86.1, which is a vector equation

$$\frac{\partial u_i}{\partial t} + (u \cdot \nabla)u_i - \nu\Delta u_i + \frac{\partial p}{\partial x_i} = f_i, \qquad i = 1, 2, 3,$$

is the *momentum equation* expressing Newton's second law stating that the acceleration is proportional to the force, and the second equation expresses the incompressibility condition. We consider here the case of Dirichlet boundary conditions with the velocity u being prescribed on the boundary Γ. Below we consider Neumann and Robin boundary conditions. Below we will often write for short $(u \cdot \nabla)u = u \cdot \nabla u$.

The linear *Stokes equations* are obtained omitting the nonlinear term $u \cdot \nabla u$, which is possible if the velocity u is small, corresponding to *creeping flow.*

The *Reynolds number Re* is defined by $Re = \frac{uL}{\nu}$, where u represents a velocity and and L a length scale characteristic of the flow. The size of the Reynolds number is decisive. If $Re \sim 1$, then the flow is very viscous, a situation met in e.g. polymer flow or forming processes. In most applications in areo/hydro-dynamics, Re is much larger than 1, often very large up to 10^6 or even larger. In these cases with small viscosity, the flow may be very complex or turbulent.

There is a stationary analog of 86.1 assuming the solution to be independent of time along with the driving force and boundary data. A stationary solution normally arises as a limit of a time-dependent solution as time tends to infinity, and this is often reflected in the computation of a stationary solution through some kind of time-stepping until convergence. For larger Reynolds numbers, stable stationary solutions in general do not not exist.

86.3 Numerical methods for the incompressible Navier-Stokes equations

Trying to solve the incompressible Navier-Stokes equations numerically, we meet the following difficulties

- instabilities from discretization of convection terms,
- pressure instabilities in equal order interpolation of velocity and pressure.

The simplest cure to convection instability is to increase the viscosity ν in the computation so that $\nu \geq uh$, where u is the local fluid velocity and h is the local mesh size. The simplest stabilization of the pressure p, is to modify the incompressibility equation $\nabla \cdot u = 0$ to $-\nabla \cdot (\delta \nabla p) + \nabla \cdot u = 0$, with $\delta \approx h^2$ with $h(x)$ the local mesh size.

In Galerkin methods the stabilization can be achieved in higher-order consistent form by adding least-squares control of residuals. We present this approach below in the context of the cG(1)dG(0) method with cG(1) in space and dG(0) in time. We also present corresponding cG(1)cG(1) and cG(1)dG(1) methods.

86.4 The cG(1)dG(0) method

We now present the cG(1)dG(0) method for 86.1 starting with the case of homogeneous Dirichlet boundary conditions. Let $0 = t_0 < t_1 < ... < t_N = T$ be a sequence of discrete time levels with associated time steps $k_n = t_n - t_{n-1}$. Let W_h be the usual finite element space of continuous piecewise linear functions on a triangulation $\mathcal{T}_h = \{K\}$ of Ω with mesh function $h(x)$. Let W_h^0 be the space of functions in W_h vanishing on Γ. We shall seek an approximate velocity $U(x,t)$ such that $U(x,t)$ is continuous and piecewise linear in x for each t, and $U(x,t)$ is piecewise constant in t for each x. Similarly, we shall seek an approximate pressure $P(x,t)$ which is continuous piecewise linear in x and piecewise constant in t. More precisely, we shall seek $U^n \in V_h^0$ with $V_h^0 = W_h^0 \times W_h^0 \times W_h^0$ and $P^n \in W_h$ for $n = 1, ..., N$, and we shall set

$$\begin{aligned} U(x,t) &= U^n(x) \quad x \in \Omega, \quad t \in (t_{n-1}, t_n], \\ P(x,t) &= P^n(x) \quad x \in \Omega, \quad t \in (t_{n-1}, t_n]. \end{aligned} \tag{86.3}$$

Further we write for velocites $v = (v_i)$ and $w = (w_i)$

$$(v,w) = \int_\Omega v \cdot w \, dx, \qquad (\nabla v, \nabla w) = \int_\Omega \sum_i^3 \nabla v_i \cdot \nabla w_i \, dx,$$

and similarly for scalar functions p and q defined on Ω:

$$(p,q) = \int_\Omega pq \, dx.$$

We now formulate the cG(1)dG(0) method without stabilization as follows: For $n = 1, ..., N$, find $(U^n, P^n) \in V_h^0 \times W_h$ such that

$$\begin{aligned} (\frac{U^n - U^{n-1}}{k_n}, v) + (U^n \cdot \nabla U^n + \nabla P^n, v) + (\nu \nabla U^n, \nabla v) = (f^n, v) \quad \forall v \in V_h^0, \\ (\nabla \cdot U^n, q) = 0 \quad \forall q \in W_h, \end{aligned} \tag{86.4}$$

where $U^0 = u^0$, and we set $f^n(x) = f(x, t_n)$. We see that the discrete equations result from multiplication of the momentum equation with $v \in V_h^0$ and the incompressibility equation by $q \in W_h$, followed by integration over Ω including integration by parts in the term $(-\nu \Delta U, v)$.

We can write the cG(1)dG(0) method without stabilization alternatively as follows: For $n = 1, ..., N$, find $(U^n, P^n) \in V_h^0 \times W_h$ such that

$$\begin{aligned} &(\frac{U^n - U^{n-1}}{k_n}, v) + (U^n \cdot \nabla U^n + \nabla P^n, v) + (\nabla \cdot U^n, q) \\ &+ (\nu \nabla U^n, \nabla v) = (f^n, v) \quad \forall (v,q) \in V_h^0 \times W_h, \end{aligned} \tag{86.5}$$

where we simply added the equations in 86.4.

The cG(1)dG(0) method with stabilization takes the form: For $n = 1, ..., N$, find $(U^n, P^n) \in V_h^0 \times W_h$ such that

$$\begin{aligned}
&(\frac{U^n - U^{n-1}}{k_n}, v) + (U^n \cdot \nabla U^n + \nabla P^n, v + \delta(U^n \cdot \nabla v + \nabla q)) + (\nabla \cdot U^n, q) \\
&+ (\nu \nabla U^n, \nabla v) = (f^n, v + \delta(U^n \cdot \nabla v + \nabla q)) \quad \forall (v, q) \in V_h^0 \times W_h,
\end{aligned} \tag{86.6}$$

where δ is a stabilization parameter defined as follows: $\delta(x) = h^2(x)$ in the case of *diffusion-dominated* flow with $\nu \geq Uh$, and

$$\delta = (\frac{1}{k} + \frac{U}{h})^{-1} \tag{86.7}$$

in the case of *convection dominated* flow with $\nu < Uh$. Note that if $k \approx \frac{h}{U}$, which is a natural choice of time step in the convection-dominated case, then $\delta \approx \frac{1}{2}\frac{h}{U}$. Note further that the stabilized form 86.6 of the cG(1)dG(0) method is obtained by replacing v by $v + \delta(U^n \cdot \nabla v + \nabla q)$ in the terms $(U^n \cdot \nabla U^n + \nabla P^n, v)$ and (f^n, v). In principle, we should make the replacement throughtout, but in the present case of the cG(1)dG(0), only the indicated terms get involved because of the low order of the approximations. The perturbation in the stabilized method is of size δ, and thus the stabilized method has the same order as the original method (first order in h if $k \sim h$).

Letting v vary in 86.6 while choosing $q = 0$, we get the following equation (the discrete momentum equation):

$$\begin{aligned}
&(\frac{U^n - U^{n-1}}{k_n}, v) + (U^n \cdot \nabla U^n + \nabla P^n, v + \delta U^n \cdot \nabla v) \\
&+ (\nu \nabla U^n, \nabla v) = (f^n, v + \delta U^n \cdot \nabla v) \quad \forall v \in V_h^0,
\end{aligned} \tag{86.8}$$

and letting q vary while setting $v = 0$, we get the following discrete pressure equation:

$$(\delta \nabla P^n, \nabla q) = -(\delta U^n \cdot \nabla U^n, \nabla q) - (\nabla \cdot U^n, q) + (\delta f^n, \nabla q) \quad \forall q \in W_h. \tag{86.9}$$

We normally seek to solve the system 86.6 iteratively alternatively solving the velocity equation 86.8 for U^n with P^n given, and the pressure equation 86.9 for P^n with U^n given.

86.5 The cG(1)cG(1) method

We present the following cG(1)cG(1) variant of the cG(1)dG(0) method with cG(1) in time instead of dG(0): For $n = 1, ..., N$, find $(U^n, P^n) \in V_h^0 \times W_h$ such that

$$
\begin{aligned}
&(\frac{U^n - U^{n-1}}{k_n}, v) + (\hat{U}^n \cdot \nabla \hat{U}^n + \nabla P^n, v + \delta(\hat{U}^n \cdot \nabla v + \nabla q)) + (\nabla \cdot \hat{U}^n, q) \\
&+ (\nu \nabla \hat{U}^n, \nabla v) = (f^n, v + \delta(\hat{U}^n \cdot \nabla v + \nabla q)) \quad \forall (v, q) \in V_h^0 \times W_h,
\end{aligned}
\tag{86.10}
$$

where $\hat{U}^n = \frac{1}{2}(U^n + U^{n-1})$. Evidently, we obtained the cG(1) version by changing from U^n to $\hat{U}^n$ in all terms but the first in the cG(1)dG(0) method.

86.6 The cG(1)dG(1) method

We shall now formulate the cG(1)dG(1) method obtained by replacing dG(0) by dG(1) in the cG(1)dG(0) method. In this method the discrete velocity $U(x,t)$ is piecewise linear linear in time on each time interval I_n, with possibly discontinuitites at the discrete time levels t_n. More precisely, we make the Ansatz:

$$
U^n(x,t) = \frac{t_n - t}{k_n} U_+^{n-1}(x) + \frac{t - t_{n-1}}{k_n} U_-^n(x), \qquad \text{for } t_{n-1} < t < t_n,
\tag{86.11}
$$

where U_+^{n-1} and U_-^n belong to V_h^0. We note that

$$
U_\pm^n(x) = \lim_{s \to 0^+} U(x, t_n \pm s)
$$

is the limit of $U(x,t)$ as t approaches t_n from below $(-)$, or above $(+)$. The cG(1)dG(1) method takes the form: For $n = 1, ..., N$, find U^n of the form 86.11 and $P^n \in W_h$, such that for all $v(x,t) = w_1(x,t) + (t - t_{n-1}) w_2(x,t)$ with $w_1, w_2 \in V_h^0$ and $q \in W_h$,

$$
\begin{aligned}
&(U_+^{n-1} - U_-^{n-1}, v) \\
&+ \int_{t_{n-1}}^{t_n} ((\dot{U}^n + U^n \cdot \nabla U^n + \nabla P^n, v + \delta(\dot{U}^n + U^n \cdot \nabla v + \nabla q)) + (\nabla \cdot U^n, q))\, dt \\
&+ \int_{t_{n-1}}^{t_n} (\nu \nabla U^n, \nabla v)\, dt = \int_{t_{n-1}}^{t_n} (f^n, v + \delta(\dot{U} + U^n \cdot \nabla v + \nabla q)).
\end{aligned}
\tag{86.12}
$$

We may similary let P be piecewise linear discontinuous in time.

86.7 Neumann boundary conditions

To properly model Neumann boundary conditions, we first need to recall that the components σ_{ij} of the *total stress tensor* $\sigma = (\sigma_{ij})$ acting on a fluid element, are given by

$$\sigma_{ij} = \bar{\sigma}_{ij} - p\delta_{ij}, \quad i,j = 1,2,3,$$

where the *stress deviatoric* $\bar{\sigma} = (\bar{\sigma}_{ij})$ is coupled to the *strain tensor* $\epsilon(u) = (\epsilon_{ij}(u))$ with components

$$\epsilon_{ij}(u) = (\partial u_i/\partial x_j + \partial u_j/\partial x_i)/2, \quad i,j = 1,2,3,$$

through the constitutive relation of a *Newtonian fluid*:

$$\bar{\sigma}_{ij} = 2\nu\epsilon_{ij}(u), \quad i,j = 1,2,3,$$

where ν is the constant viscosity, and $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. We observe that the trace of the stress deviatoric is zero, that is,

$$\sum_{i=1}^{3} \bar{\sigma}_{ii} = 2\nu \sum_{i=1}^{3} \epsilon_{ii}(u) = 2\nu \nabla \cdot u = 0,$$

and thus the total stress σ is decomposed into a stress deviatoric $\bar{\sigma}$ with zero trace and an isotropic pressure p. Further, a direct computation shows that

$$\nu\Delta u - \nabla p = \nabla \cdot \sigma, \tag{86.13}$$

where $\nabla \cdot \sigma$ is a vector with components $(\nabla \cdot \sigma)_i$ given by

$$(\nabla \cdot \sigma)_i = \sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j}.$$

Multiplying 86.13 by $v = (v_i)$ with $v = 0$ on Γ and integrating by parts, we find that

$$\nu(\nabla u, \nabla v) + (\nabla p, v) = 2\nu(\epsilon(u), \epsilon(v)) + (\nabla p, v),$$

where

$$(\epsilon(u), \epsilon(v)) = \sum_{i,j=1}^{3} \int_{\Omega} \epsilon_{ij}(u)\epsilon_{ij}(v)\, dx.$$

We are thus led to replace the term $(\nu\nabla u, \nabla v)$ by the term $(2\nu\epsilon(u), \epsilon(v))$ in variational formulations of the Navier-Stokes equations. In the case of Dirichlet boundary conditions for the velocity the two expressions are equal, since the test velocity v vanishes on Γ, but in the case of Neumann type boundary conditions the replacement opens the possibility of enforcing in variational form a Neuman boundary condition of the form

$$\sum_{j=1}^{3} \sigma_{ij} n_j = \sum_{j=1}^{3} \bar{\sigma}_{ij} n_j - p n_i = \sum_{j=1}^{3} 2\nu\epsilon_{ij}(u) n_j - p n_i = g_i \quad \text{on } \Gamma_2, \quad i = 1,2,3, \tag{86.14}$$

which expresses that the total force on the boundary part Γ_2 is equal to the given force $g = (g_i)$. For example, if $g = 0$, then this condition expresses that the total force is zero on Γ_2, which we may use as an outflow boundary condition simulating that the fluid freely flows out into a large reservoir. More precisely, the presence of the terms

$$-(p, \nabla \cdot v) + (2\nu\epsilon(u), \epsilon(v))$$

in a variational formulation with v varying freely on Γ_2, will enforce a homogeneous Neumann boundary condition 86.14 upon integration by parts.

We now consider a typical situation with the boundary Γ decomposed into two parts Γ_1 an Γ_2 with the velocity being equal to a given velocity w on Γ_1 and imposing the homogeneous Neumann condition 86.14 on Γ_2. For simplicity, we assume that w is independent of time, the extension to time dependence of w being evident. Typically, w will be zero on a part of Γ_1 and will be directed into Ω on the remaining part corresponding to a given inflow.

We let V_h be the space of continuous piecewise linear velocities v on a triangulation $\mathcal{T}_h = \{K\}$ of Ω with mesh function $h(x)$, satisfying the boundary condition $v = w$ on Γ_1, and let V_h^0 be the corresponding test space of functions with $v = 0$ on Γ_1. Let W_h be the space of continuous piecewise linear pressures p on $\mathcal{T}_h = \{K\}$, and W_h^0 the corresponding test space of pressures q such that $q = 0$ on Γ_2.

The stabilized cG(1)dG(0) method can be formulated as follows: For $n = 1, ..., N$ seek $U^n \in V_h$ and $P^n \in W_h$ such that

$$\begin{aligned}(\frac{U^n - U^{n-1}}{k_n}, v) + (U^n \cdot \nabla U^n, v + \delta U^n \cdot \nabla v) - (P^n, \nabla \cdot v) \\ + (2\nu\epsilon(U^n), \epsilon(v)) = (f^n, v + \delta U^n \cdot \nabla v) \quad \forall v \in V_h^0,\end{aligned} \tag{86.15}$$

$$(\delta\nabla P^n, \nabla q) = -(\delta U^n \cdot \nabla U^n, \nabla q) - (\nabla \cdot U^n, q) + (\delta f^n, \nabla q) \quad \forall q \in W_h^0, \tag{86.16}$$

where we choose P^n on Γ_2 according to 86.14 with $g = 0$ and u replaced by U. Again we seek to solve the system iteratively alternatively solving the velocity equation 86.15 for U^n with P^n given, and the pressure equation 86.16 for P^n with U^n given.

For a discussion of artificial outflow boundary conditions which may be used to truncate the domain of computation in a flow simulation, see the survey article by Rannacher.

86.8 Periodic boundary conditions

Another possibility is to truncate the computational domain by using periodic boundary conditions, where the nodes in two parts of the boundary are identified so that what goes out through one part of the boundary enters the domain through the corresponding other part of the boundary. This corresponds to the assumption that the flow in the computational domain is repeated periodically throughout the physical flow domain.

86.9 Computational examples

We now present some computational examples of 3d time dependent flows, using the stabilized cG(1)cG(1) method on a mesh with meshsize $h = 1/32$.

In Figure 86.1 we present the solution of a bluff body problem: a flow in a channel with 1x1 square cross section and length 4, with a square obstacle with side length 0.25 centered at $(0.5, 0.5, 0.5)$. We have used zero Dirichlet boundary condition for the velocity on the side walls and Neumann outflow boundary conditions on the outflow boundary. On the inflow a parabolic velocity is prescribed.

In Figure 86.2 we present the solution of a step down problem in a similar channel with a step down of height and length 0.5.

Finally in Figure 86.3 we present computations of transition to turbulence in a circular jet flow with streamwise velocity 1 in the jet and zero outside the jet, where we apply a small random perturbation. Here we have used periodic boundary conditions in all directions.

For more details on these computations see [**?**] and [**?**].

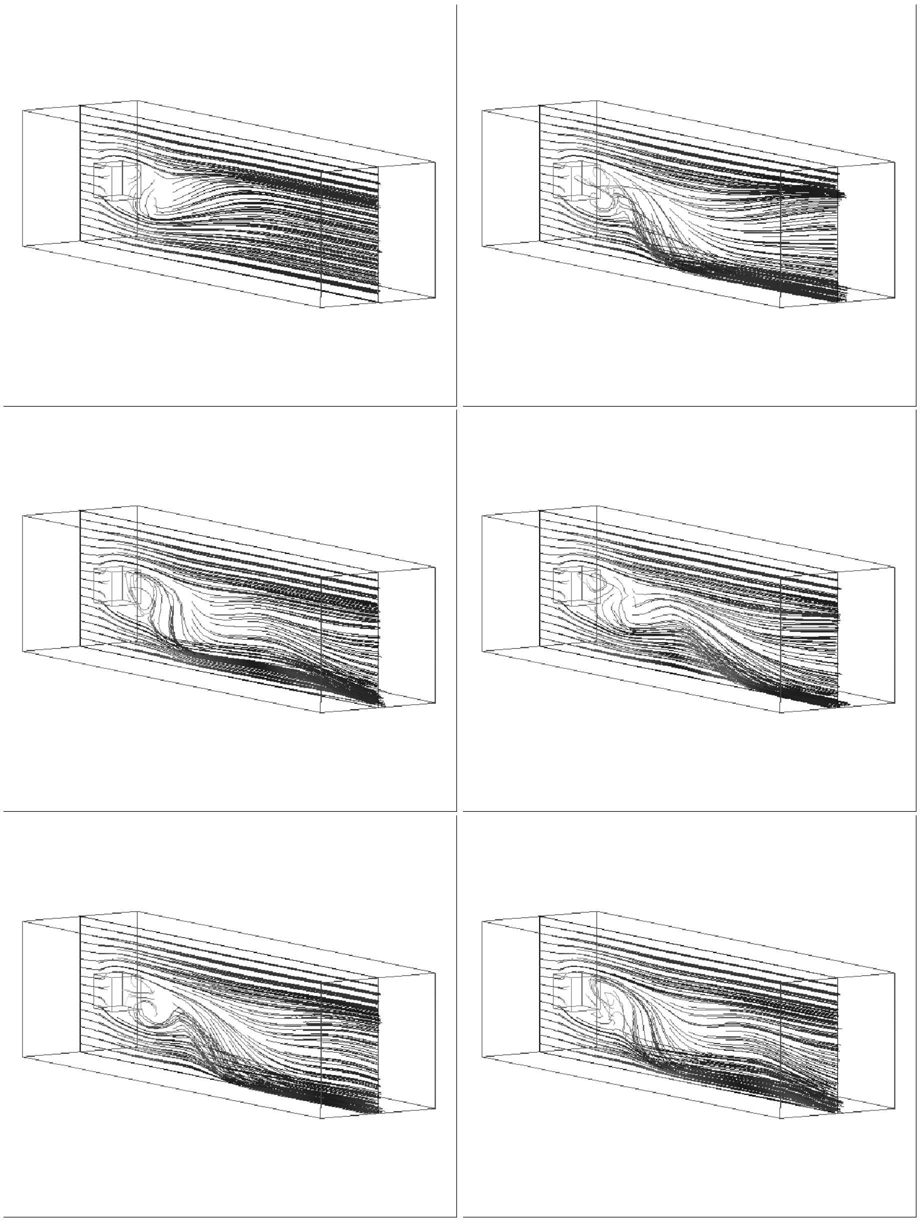

FIGURE 86.1. Bluff body flow computations for $t = 2, 4, 6, 8, 10, 12$.

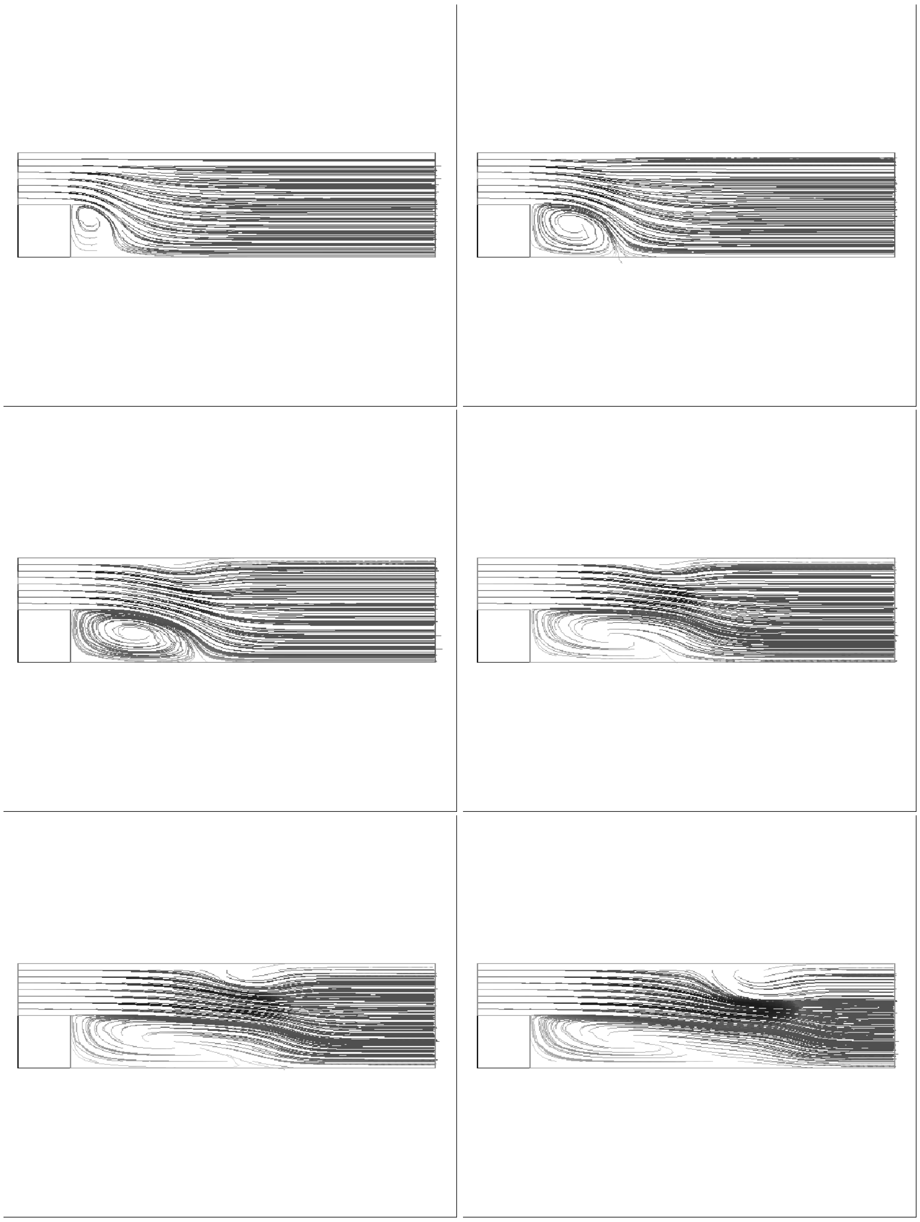

FIGURE 86.2. Step down flow computations for $t = 1, 2, 3, 4, 5, 6$.

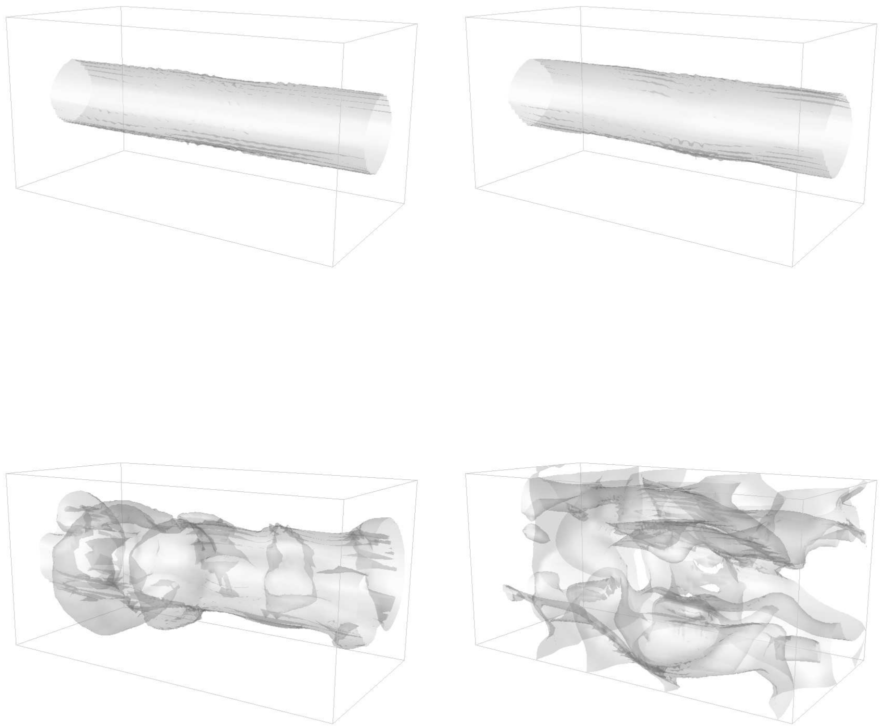

FIGURE 86.3. Streamwise velocity isosurfaces for $|u_1| = 0.02$ in jet flow for $t = 5, 7, 10, 15$