TMA225 Differential Equations and Scientific Computing, part A

Solutions to Problems Week 1 – 7

Note: Now complete except for the *-problems Week 7.

May 20, 2003

Contents

Week 1:

Problem 1. Use the expressions $\lambda_a(x) = \frac{b-x}{b-a}$ and $\lambda_b(x) = \frac{x-a}{b-a}$ to show that

$$\lambda_a(x) + \lambda_b(x) = 1; \quad a\,\lambda_a(x) + b\,\lambda_b(x) = x.$$

Give a geometrical interpretation by plotting $\lambda_a(x)$, $\lambda_b(x)$, $\lambda_a(x) + \lambda_b(x)$, $a\,\lambda_a(x)$, $b\,\lambda_b(x)$, $a\,\lambda_a(x) + b\,\lambda_b(x)$ in the same figure.

Solution: Direct calculation gives $\lambda_a(x) + \lambda_b(x) = \frac{b-x}{b-a} + \frac{x-a}{b-a} = 1$ and $a\,\lambda_a(x) + b\,\lambda_b(x) = a\,\frac{b-x}{b-a} + b\,\frac{x-a}{b-a} = x$. The functions for the case $a = 2$ and $b = 3$ are plotted in Figure 1. $\square$

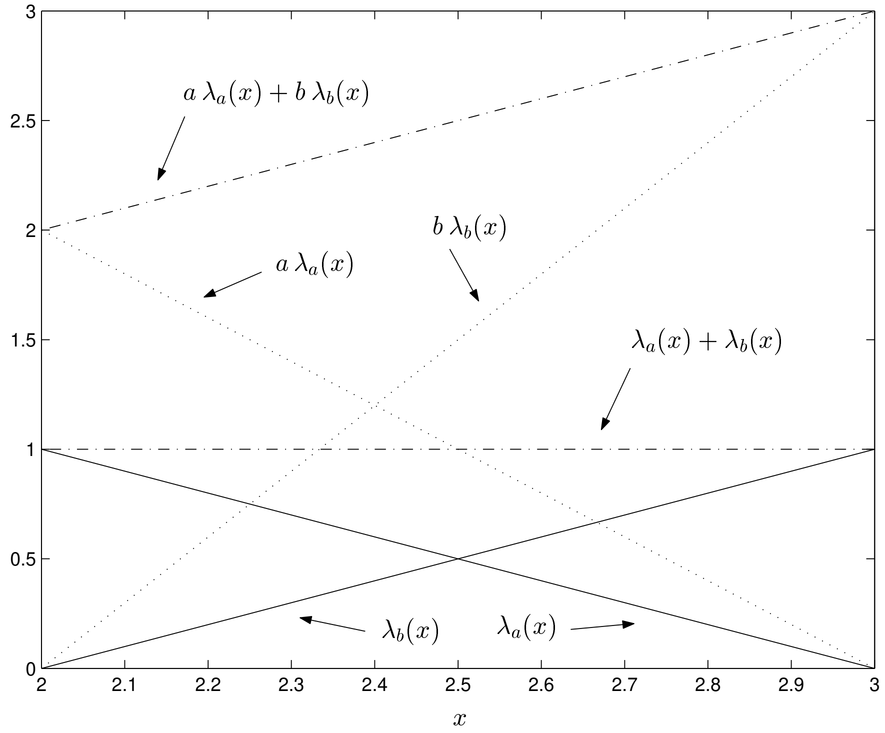

Figure 1: Problem 1 (Week 1). A plot of the functions.

Problem 2. Let $0 = x_0 < x_1 < x_2 < x_3 = 1$, where $x_1 = 1/6$ and $x_2 = 1/2$, be a partition of the interval $[0, 1]$ into three subintervals.

(a) Determine analytical expressions for the "hat-functions" φ_0, φ_1, φ_2, φ_3 in V_h (the space of continuous piecewise linear functions on this partition). Draw a figure.

(b) Which is the dimension of V_h?

(c) Plot the mesh function $h(x)$.

Solution:

(a) The "hat-functions" are given by the formula (with obvious modifications for φ_0 and φ_3):

$$\varphi_i(x) = \begin{cases} 0, & x \notin [x_{i-1}, x_{i+1}] \\ \frac{x - x_{i-1}}{x_i - x_{i-1}}, & x \in [x_{i-1}, x_i] \\ \frac{x_{i+1} - x}{x_{i+1} - x_i}, & x \in [x_i, x_{i+1}] \end{cases}$$

This gives

$$\varphi_0(x) = \begin{cases} 0, & x \notin [x_0, x_1] \\ 1 - 6x, & x \in [x_0, x_1] \end{cases} , \quad \varphi_1(x) = \begin{cases} 0, & x \notin [x_0, x_2] \\ 6x, & x \in [x_0, x_1] \\ \frac{3-6x}{2}, & x \in [x_1, x_2] \end{cases}$$

and

$$\varphi_2(x) = \begin{cases} 0, & x \notin [x_1, x_3] \\ \frac{6x-1}{2}, & x \in [x_1, x_2] \\ 2 - 2x, & x \in [x_2, x_3] \end{cases} , \quad \varphi_3(x) = \begin{cases} 0, & x \notin [x_2, x_3] \\ 2x - 1, & x \in [x_2, x_3] \end{cases} ,$$

where $x_0 = 0$, $x_1 = 1/6$, $x_2 = 1/2$ and $x_3 = 1$. See Figure 2.

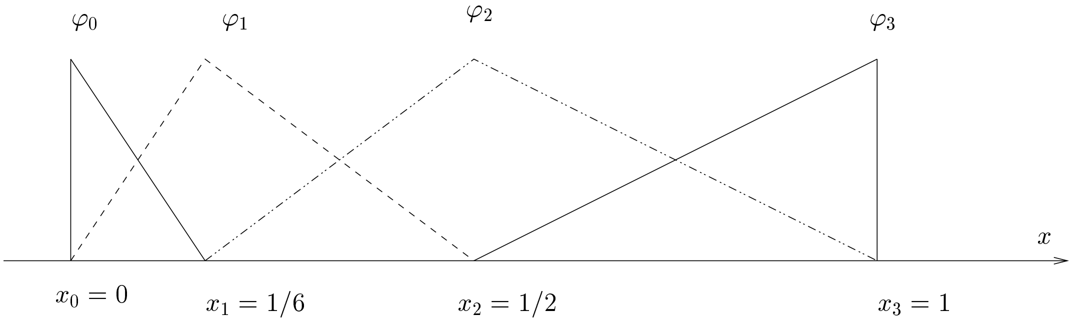

Figure 2: Problem 2(a) (Week 1). A plot of the "hat-functions".

(b) The dimension of V_h is equal to the number of basis functions which in this case is 4.
(c) See Figure 3. $\square$

Problem 3. Let $f : [0,\, 1] \to \boldsymbol{R}$ be a Lipschitz continuous function. Determine the linear interpolant $\pi f \in \mathcal{P}(0,\, 1)$ and plot f and πf in the same figure, when
(a) $f(x) = x^2$,
(b) $f(x) = \sin(\pi x)$.
Solution: In general, the linear (nodal) interpolant $\pi f \in \mathcal{P}(x_0,\, x_1)$ can be written as

$$\pi f(x) = f(x_0)\, \varphi_0(x) + f(x_1)\, \varphi_1(x),$$

where $\varphi_i(x)$ form a basis of the space $\mathcal{P}(x_0,\, x_1)$ of linear polynomials on $I = [x_0,\, x_1]$. The x_i's are *nodes* where the interpolant's value is the same as the function's value. The basis functions we use are the "hat functions" $\varphi_i(x)$, $i = 0, 1$. Remember that $\varphi_0(x) = 1 - x$ and $\varphi_1(x) = x$, on $I = [0, 1]$.

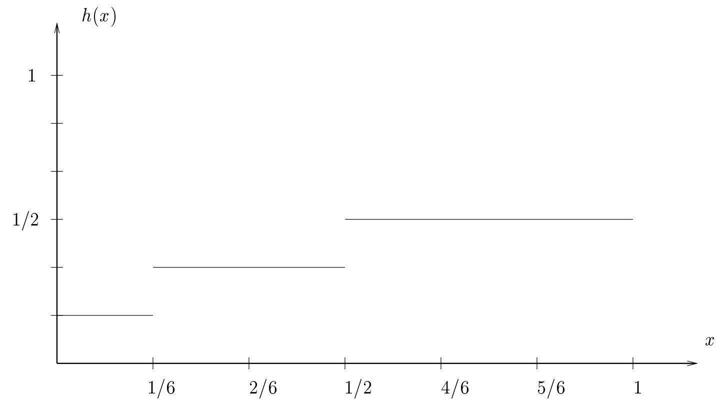

Figure 3: Problem 2(c) (Week 1). A plot of the mesh function.

(a) Therefore, the linear (nodal) interpolant of $f(x) = x^2$ on I can be written as

$$\begin{aligned}
\pi f(x) &= \sum_{i=1}^{2} f(x_i)\varphi_i(x) \\
&= f(0)\,\varphi_0(x) + f(1)\,\varphi_1(x) \\
&= 0 \cdot (1 - x) + 1 \cdot x \\
&= x
\end{aligned}$$

(b) With $f(x) = \sin(\pi x)$, we analogously get

$$\begin{aligned}
\pi f(x) &= \sum_{i=1}^{2} f(x_i)\varphi_i(x) \\
&= f(0)\,\varphi_0(x) + f(1)\,\varphi_1(x) \\
&= \sin(0)\,(1 - x) + \sin(\pi)\,x \\
&= 0
\end{aligned}$$

Obviously, the interpolant we've computed here is a poor approximation of $f(x) = \sin(\pi x)$. $\square$

Problem 4. Let $f : [0, 1] \to \boldsymbol{R}$ be a Lipschitz continuous function. Determine the continuous piecewise linear interpolant $\pi_h f \in V_h$, with $h(x)$ and V_h as in *Problem* 2 *(Week* 1*)*, and plot f and $\pi_h f$ in the same figure, when
(a) $f(x) = x^2$,
(b) $f(x) = \sin(\pi x)$.
Have we chosen a proper partition to approximate these functions? Can you think of a better one in case (a) and (b) if we are restricted to three subintervals?
Solution: The interval is partitioned according to $0 = x_0 < x_1 < x_2 < x_3 = 1$, with $x_1 = 1/6$ and $x_2 = 1/2$. On each subinterval we want the approximation, $\pi_h f(x)$, of $f(x)$

to be a straight line, $\alpha x + \beta$, and we want $\pi_h f(x)$ to *interpolate* $f(x)$ at the nodes $\{x_i\}_{i=0}^3$, i.e., $\pi_h f(x_i) = f(x_i)$, $i = 0, \ldots, 3$. (Note that this makes $\pi_h f(x)$ continuous also at the node-points.) This is accomplished by defining

$$\pi_h f(x) = f(x_0)\,\varphi_0(x) + f(x_1)\,\varphi_1(x) + f(x_2)\,\varphi_2(x) + f(x_3)\,\varphi_3(x)$$

where the "hat-functions" $\{\varphi_i\}_{i=0}^3$ have been computed in *Problem 2 (Week 1)*.
(a) With $f(x) = x^2$ we get

$$\begin{aligned}
\pi_h f(x) &= f(x_0)\,\varphi_0(x) + f(x_1)\,\varphi_1(x) + f(x_2)\,\varphi_2(x) + f(x_3)\,\varphi_3(x) \\
&= 0 \cdot \varphi_0(x) + \frac{1}{36} \cdot \varphi_1(x) + \frac{1}{4} \cdot \varphi_2(x) + 1 \cdot \varphi_3(x)
\end{aligned}$$

$$\left(\begin{aligned}
&= \begin{cases} 0 \cdot (1-6x) + \frac{1}{36} \cdot 6x, & x \in [0,\, 1/6] \\ \frac{1}{36} \cdot (\frac{3}{2} - 3x) + \frac{1}{4} \cdot (3x - \frac{1}{2}), & x \in [1/6,\, 1/2] \\ \frac{1}{4} \cdot (2-2x) + 1 \cdot (2x-1), & x \in [1/2,\, 1] \end{cases} \\
&= \begin{cases} x/6, & x \in [0,\, 1/6] \\ -1/12 + 2x/3, & x \in [1/6,\, 1/2] \\ -1/2 + 3x/2, & x \in [1/2,\, 1] \end{cases}
\end{aligned} \right)$$

Remark. We are usually content with writing a function $v \in V_h$ in the form $v(x) = c_0\,\varphi_0(x) + c_1\,\varphi_1(x) + c_2\,\varphi_2(x) + c_3\,\varphi_3(x)$ which tells us that the nodal values are $(c_0,\, c_1,\, c_2,\, c_3)$ and that v is linear in between. Compare with plotting v in Matlab: `>> plot([x0 x1 x2 x3], [c0 c1 c2 c3])` connects the four points $(x_i,\, c_i)$ with straight lines. If we for some reason need to know the analytical expressions on each subinterval, they may of course be computed as above.

(b) For $f(x) = \sin(\pi x)$ we similarly get

$$\begin{aligned}
\pi_h f(x) &= f(x_0)\,\varphi_0(x) + f(x_1)\,\varphi_1(x) + f(x_2)\,\varphi_2(x) + f(x_3)\,\varphi_3(x) \\
&= 0 \cdot \varphi_0(x) + \frac{1}{2} \cdot \varphi_1(x) + 1 \cdot \varphi_2(x) + 0 \cdot \varphi_3(x)
\end{aligned}$$

$$\left(\begin{aligned}
&= \begin{cases} 0 \cdot (1-6x) + \frac{1}{2} \cdot 6x, & x \in [0,\, 1/6] \\ \frac{1}{2} \cdot (\frac{3}{2} - 3x) + 1 \cdot (3x - \frac{1}{2}), & x \in [1/6,\, 1/2] \\ 1 \cdot (2-2x) + 0 \cdot (2x-1), & x \in [1/2,\, 1] \end{cases} \\
&= \begin{cases} 3x, & x \in [0,\, 1/6] \\ 3x/2 + 1/4, & x \in [1/6,\, 1/2] \\ 2 - 2x, & x \in [1/2,\, 1] \end{cases}
\end{aligned} \right)$$

□

Problem 5. Let $h(x)$ be the mesh function for the partition defined in *Problem* 2 *(Week 1)*. Compute $||f - \pi_h f||_{L_\infty(0,\,1)}$ and $\frac{1}{8}||h^2 f''||_{L_\infty(0,\,1)}$, when

(a) $f(x) = x^2$,
(b) $f(x) = \sin(\pi x)$.
Hint: To compute $||f - \pi_h f||_{L_\infty(0,1)}$ you need to maximize the function $g(x) = |f(x) - \pi_h f(x)|$ on each subinterval and choose the largest of these three maxima. You can control your answers by also doing the computations with *Piecewise Polynomial Lab*.

If you think you have found better partitions in the end of *Problem* 4 *(Week* 1*)*, repeat the computations for these. Utilize *Piecewise Polynomial Lab*!
Solution: We have the mesh function

$$h(x) = \begin{cases} 1/6, & 0 < x < 1/6 \\ 1/3, & 1/6 < x < 1/2, \\ 1/2, & 1/2 < x < 1, \end{cases}$$

and want to compute $||f - \pi_h f||_{L_\infty(0,1)} = \max_{x \in [0,1]} |f(x) - \pi_h f(x)|$ and $\frac{1}{8}||h^2 f''||_{L_\infty(0,1)} = \frac{1}{8}\max_{x \in [0,1]} |h(x)^2 f''(x)|$.
(a) From *Problem* 4*(a) (Week* 1*)*, we have

$$|f(x) - \pi_h f(x)| = \begin{cases} |x^2 - x/6|, & 0 \le x \le 1/6, \\ |x^2 - (2x/3 - 1/12)\,|, & 1/6 \le x \le 1/2, \\ |x^2 - (3x/2 - 1/2)\,|, & 1/2 \le x \le 1. \end{cases}$$

Find maxima for each subinterval: (*Note:* Since $f''(x) = 2 > 0$, $f(x)$ is a *convex* function and the interpolant $\pi_h f(x)$ will therefore always be greater than $f(x)$. Further, since $f(x) - \pi_h f(x) = 0$ at the nodes, the local maxima will occur in the interior of the subintervals.)
$0 \le x \le 1/6:$

$$g(x) = |\underbrace{x^2 - \frac{x}{6}}_{\le 0,\ x \in [0,1/6]}| = \frac{x}{6} - x^2;$$

$$g'(x) = \frac{1}{6} - 2x = 0 \quad \Rightarrow \quad x = \frac{1}{12}; \qquad g''(x) = -2 \quad \Rightarrow \quad \textit{Maximum:}\ g(\frac{1}{12}) = \frac{1}{144}.$$

$1/6 \le x \le 1/2:$

$$g(x) = |\underbrace{x^2 - \left(\frac{2x}{3} - \frac{1}{12}\right)}_{\le 0,\ x \in [1/6,1/2]}| = \frac{2x}{3} - \frac{1}{12} - x^2;$$

$$g'(x) = \frac{2}{3} - 2x = 0 \quad \Rightarrow \quad x = \frac{1}{3}; \qquad g''(x) = -2 \quad \Rightarrow \quad \textit{Maximum:}\ g(\frac{1}{3}) = \frac{1}{36}.$$

$1/2 \le x \le 1:$

$$g(x) = |\underbrace{x^2 - \left(\frac{3x}{2} - \frac{1}{2}\right)}_{\le 0,\ x \in [1/2,1]}| = \frac{3x}{2} - \frac{1}{2} - x^2;$$

$$g'(x) = \frac{3}{2} - 2x = 0 \quad \Rightarrow \quad x = \frac{3}{4}; \qquad g''(x) = -2 \quad \Rightarrow \quad \textit{Maximum:}\ g(\frac{3}{4}) = \frac{1}{16}.$$

By comparing the three local maxima we get:

$$||f - \pi_h f||_{L_\infty(0,\,1)} = \frac{1}{16}.$$

Since

$$\frac{1}{8}|h(x)^2 f''(x)| = \begin{cases} \frac{2}{8\cdot 36} = \frac{1}{144}, & 0 < x < 1/6, \\ \frac{2}{8\cdot 9} = \frac{1}{36}, & 1/6 < x < 1/2, \\ \frac{2}{8\cdot 4} = \frac{1}{16}, & 1/2 < x < 1, \end{cases}$$

we also get

$$\frac{1}{8}||h^2 f''||_{L_\infty(0,\,1)} = \frac{1}{16}.$$

Remark. The reason we have equality in this case is that $f''(x) = 2$ is *constant.*

(b) From *Problem* 4*(b) (Week* 1*)*, we have

$$|f(x) - \pi_h f(x)| = \begin{cases} |\sin(\pi x) - 3x|, & 0 \le x \le 1/6, \\ |\sin(\pi x) - (3x/2 + 1/4)\,|, & 1/6 \le x \le 1/2, \\ |\sin(\pi x) - (2 - 2x)\,|, & 1/2 \le x \le 1. \end{cases}$$

Find maxima for each subinterval: (*Note:* Since $f''(x) = -\pi^2 \sin(\pi x) < 0$, for $x \in (0,\,1)$, $f(x)$ is *concave* on this interval and the interpolant $\pi_h f(x)$ will therefore be lesser than $f(x)$.)

$0 \le x \le 1/6:$

$$g(x) = |\underbrace{\sin(\pi x) - 3x}_{\ge 0,\; x \in [0,\,1/6]}| = \sin(\pi x) - 3x;$$

$$g'(x) = \pi\cos(\pi x) - 3 = 0 \quad \Rightarrow \quad x = \frac{1}{\pi}\arccos(\frac{3}{\pi}) \approx 0.096 \in [0,\ 1/6];$$

$$g''(x) = -\pi^2 \sin(\pi x) < 0 \quad \Rightarrow \quad \textit{Maximum: } g(\frac{1}{\pi}\arccos(\frac{3}{\pi})) \approx 0.009.$$

$1/6 \le x \le 1/2:$

$$g(x) = |\underbrace{\sin(\pi x) - \left(\frac{3x}{2} + \frac{1}{4}\right)}_{\ge 0,\; x \in [1/6,\,1/2]}| = \sin(\pi x) - \left(\frac{3x}{2} + \frac{1}{4}\right);$$

$$g'(x) = \pi\cos(\pi x) - \frac{3}{2} = 0 \quad \Rightarrow \quad x = \frac{1}{\pi}\arccos(\frac{3}{2\pi}) \approx 0.342 \in [1/6,\ 1/2];$$

$$g''(x) = -\pi^2 \sin(\pi x) < 0 \quad \Rightarrow \quad \textit{Maximum: } g(\frac{1}{\pi}\arccos(\frac{3}{2\pi})) \approx 0.116.$$

$1/2 \le x \le 1:$

$$g(x) = |\underbrace{\sin(\pi x) - (2 - 2x)}_{\ge 0,\; x \in [1/2,\,1]}| = \sin(\pi x) - (2 - 2x)\,;$$

$$g'(x) = \pi \cos(\pi x) + 2 = 0 \quad \Rightarrow \quad x = \frac{1}{\pi} \arccos(-\frac{2}{\pi}) \approx 0.720 \in [1/2,\, 1];$$

$$g''(x) = -\pi^2 \sin(\pi x) < 0 \quad \Rightarrow \quad \textit{Maximum:}\ g(\frac{1}{\pi} \arccos(-\frac{2}{\pi})) \approx 0.211.$$

By comparing the three local maxima we get:

$$||f - \pi_h f||_{L_\infty(0,\,1)} = \sin(\arccos(-\frac{2}{\pi})) - 2(1 - \frac{1}{\pi} \arccos(-\frac{2}{\pi})) \approx 0.211.$$

Since

$$\frac{1}{8}|h(x)^2 f''(x)| = \begin{cases} \frac{\pi^2 \sin(\pi x)}{8 \cdot 36} \leq \frac{\pi^2 \sin(\frac{\pi}{6})}{8 \cdot 36} = \frac{\pi^2}{576}, & 0 < x < 1/6, \\ \frac{\pi^2 \sin(\pi x)}{8 \cdot 9} \leq \frac{\pi^2 \sin(\frac{\pi}{2})}{8 \cdot 9} = \frac{\pi^2}{72}, & 1/6 < x < 1/2, \\ \frac{\pi^2 \sin(\pi x)}{8 \cdot 4} \leq \frac{\pi^2 \sin(\frac{\pi}{2})}{8 \cdot 4} = \frac{\pi^2}{32}, & 1/2 < x < 1, \end{cases}$$

we get

$$\frac{1}{8}||h^2 f''||_{L_\infty(0,\,1)} = \frac{\pi^2}{32} \approx 0.308.$$

$\square$

Week 2:

Problem 1. Let $I = (0, 1)$ and $f(x) = x^2$ for $x \in I$.
(a) Compute (analytically) $\int_I f(x)\,dx$.
(b) Compute an approximation of $\int_I f(x)\,dx$ by using the *trapezoidal rule* on the single interval $(0, 1)$.
(c) Compute an approximation of $\int_I f(x)\,dx$ by using the *mid-point rule* on the single interval $(0, 1)$.
(d) Compute the errors in (b) and (c). Compare with theory.
(e) Divide I into two subintervals of equal length. Compute an approximation of $\int_I f(x)\,dx$ by using the *trapezoidal rule* on each subinterval.
(f) Compute an approximation of $\int_I f(x)\,dx$ by using the *mid-point rule* on each subinterval.
(g) Compute the errors in (e) and (f), and compare with the errors in (b) and (c) respectively. By what factor has the error decreased?
Solution:
(a)

$$\int_0^1 x^2\,dx = \frac{1}{3}$$

(b)

$$\int_0^1 x^2\,dx \approx \frac{0^2 + 1^2}{2} = \frac{1}{2}$$

(c)

$$\int_0^1 x^2\,dx \approx (\frac{0+1}{2})^2 = \frac{1}{4}$$

(d) The error for the trapezoidal rule is $|\frac{1}{3} - \frac{1}{2}| = \frac{1}{6}$ and the error for the mid-point rule is $|\frac{1}{3} - \frac{1}{4}| = \frac{1}{12}$. Both agree with the bounds for the error on a single interval of length h: $\frac{h^3}{12} \max_{y\in[0,1]} |f''(y)| = \frac{1}{6}$ and $\frac{h^3}{24} \max_{y\in[0,1]} |f''(y)| = \frac{1}{12}$ in *Quadrature (1D)*.

Remark. The reason that we have *equality* between the error and the error bound in this case is that $f''(y) = 2$ is *constant*.

(e)

$$\int_0^1 x^2\,dx \approx \frac{0^2 + (\frac{1}{2})^2}{4} + \frac{(\frac{1}{2})^2 + 1^2}{4} = \frac{3}{8}$$

(f)

$$\int_0^1 x^2\,dx \approx \frac{(\frac{1}{4})^2}{2} + \frac{(\frac{3}{4})^2}{2} = \frac{5}{16}$$

(g) The trapezoidal rule gives $|\frac{1}{3} - \frac{3}{8}| = \frac{1}{24}$ which means that the error decreases by a factor 4 when the mesh size decreases by a factor 2. This agrees with the *global* error bound $\frac{b-a}{12} \max_{y\in[0,1]} |h^2(y) f''(y)|$ in *Quadrature (1D)*. For the mid-point rule we get the error $|\frac{1}{3} - \frac{5}{16}| = \frac{1}{48}$ which shows a similar behaviour. $\square$

Problem 2. Let $I = (0,1)$ and $f(x) = x^4$ for $x \in I$.
(a) Compute (analytically) $\int_I f(x)\,dx$.
(b) Compute an approximation of $\int_I f(x)\,dx$ by using *Simpson's rule* on the single interval $(0,1)$.
(c) Compute the error in (b). Compare with theory.
(d) Divide I into two subintervals of equal length. Compute an approximation of $\int_I f(x)\,dx$ by using *Simpson's rule* on each subinterval.
(e) Compute the error in (d), and compare with the error in (b). By what factor has the error decreased?

Solution:

(a)

$$\int_I f(x)\,dx = \int_0^1 x^4\,dx = \frac{1}{5}$$

(b)

$$\int_I f(x)\,dx \approx \frac{f(0) + 4f(\frac{0+1}{2}) + f(1)}{6} = \frac{0 + 4(\frac{1}{2})^4 + 1}{6} = \frac{5}{24}$$

(c) $Error_1 = |\frac{1}{5} - \frac{5}{24}| = |\frac{24}{120} - \frac{25}{120}| = \frac{1}{120}$. From the theory we know that the error using *Simpson's rule* on a single interval of length h must be less than or equal to

$$\frac{h^5}{2880} \max_{y\in[0,1]} |f^{(4)}(y)| = \frac{24}{2880} = \frac{1}{120}$$

Remark. The reason that we have *equality* between the error and the error bound in this case is that $f^{(4)}(y) = 24$ is *constant.*

(d)

$$\begin{aligned}\int_I f(x)\,dx &= \int_0^{1/2} f(x)\,dx + \int_{1/2}^1 f(x)\,dx \\ &\approx \frac{f(0) + 4f(\frac{0+1/2}{2}) + f(\frac{1}{2})}{6} \cdot \frac{1}{2} + \frac{f(\frac{1}{2}) + 4f(\frac{1/2+1}{2}) + f(1)}{6} \cdot \frac{1}{2} \\ &= \frac{0 + 4(\frac{1}{4})^4 + (\frac{1}{2})^4}{12} + \frac{(\frac{1}{2})^4 + 4(\frac{3}{4})^4 + 1^4}{12} = \frac{77}{384}\end{aligned}$$

(e) $Error_2 = |\frac{1}{5} - \frac{77}{384}| = |\frac{384-5\cdot 77}{1920}| = \frac{1}{1920}$. If we compare this error to the one computed above in exercise (c):

$$\frac{Error_1}{Error_2} = \frac{\frac{1}{120}}{\frac{1}{1920}} = \frac{1920}{120} = 16,$$

we see that the error has decreased by a factor 16 when the mesh size has decreased by a factor 2! This agrees with the *global* error bound $\frac{b-a}{2880} \max_{y\in[0,1]} |h^4(y) f^{(4)}(y)|$. □

Problem 3. Let $I = (0,1)$ and $f(x) = x^2$ for $x \in I$.
(a) Let V_h be the space of linear functions on I and calculate the L^2-projection $P_h f \in V_h$ of f.

Remark. In this case $h(y) \equiv 1$ and $V_h = \mathcal{P}(0,1)$.

(b) Divide I into two subintervals of equal length and let V_h be the corresponding space of continuous piecewise linear functions. Calculate the L^2-projection $P_h f \in V_h$ of f.
(c) Illustrate your results in figures and compare with the nodal interpolant $\pi_h f$.
Solution:
(a) The L^2-projection $P_h f \in V_h$ of f is the *orthogonal projection* of f onto V_h. Therefore $f - P_h f$ must be orthogonal to all $v \in V_h$, that is

$$\int_I (f - P_h f) v \, dx = 0, \quad \forall v \in V_h,$$

but from *Problem 6 (Week 2)* this is equivalent to

$$\begin{cases} \int_I (f - P_h f)\varphi_0 \, dx = 0 \\ \int_I (f - P_h f)\varphi_1 \, dx = 0 \end{cases}$$

since the "hat functions" $\varphi_0 = 1 - x$ and $\varphi_1 = x$ are a basis for V_h.

Since $P_h f \in V_h$, we make the *Ansatz*

$$P_h f = \sum_{j=0}^{1} c_j \, \varphi_j(x),$$

and inserting this Ansatz into the orthogonality relation gives

$$\sum_{j=0}^{1} c_j \int_I \varphi_j \, \varphi_i \, dx = \int_I f \, \varphi_i \, dx, \quad i = 0, 1,$$

which is a linear system with two equations and two unknowns: c_0 and c_1. It is therefore natural to state the system in matrix form, $Mc = b$, with the mass matrix $M = (m_{ij})$, $m_{ij} = \int_I \varphi_j \varphi_i \, dx$, $c = (c_0, c_1)^t$ and $b = (b_0, b_1)^t$ where $b_i = \int_I f \varphi_i \, dx$. Now, we only have to compute these integrals and solve for c. Note that $m_{ij} = m_{ji}$ (the mass matrix is *symmetric*).

$$\begin{aligned} m_{00} &= \int_I \varphi_0 \varphi_0 \, dx \\ &= \int_0^1 (1-x)^2 \, dx \\ &= 1/3 \\ m_{10} &= \int_I \varphi_0 \varphi_1 \, dx \\ &= \int_0^1 (1-x) x \, dx \\ &= 1/6 \end{aligned}$$

$$
\begin{aligned}
m_{11} &= \int_I \varphi_1 \varphi_1 \, dx \\
&= \int_0^1 x^2 \, dx \\
&= 1/3 \\
b_0 &= \int_I f \varphi_0 \, dx \\
&= \int_0^1 x^2 (1-x) \, dx \\
&= 1/12 \\
b_1 &= \int_I f \varphi_1 \, dx \\
&= \int_0^1 x^2 \cdot x \, dx \\
&= 1/4
\end{aligned}
$$

The system of equations we have to solve is then

$$
\begin{bmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 1/12 \\ 1/4 \end{bmatrix}.
$$

Hence, $c_0 = -1/6$ and $c_1 = 5/6$, which gives $P_h f(x) = c_0 \varphi_0(x) + c_1 \varphi_1(x) = -1/6 \, \varphi_0(x) + 5/6 \, \varphi_1(x) = -1/6 \cdot (1-x) + 5/6 \cdot x = -1/6 + x$.

Remark. We could in principle use any set (pair, in this case) of basis functions, for instance $\{1, x\} \subset V_h$. This choice would lead to the orthogonality relation

$$
\begin{cases} \int_I (f - P_h f) \cdot 1 \, dx = 0 \\ \int_I (f - P_h f) \cdot x \, dx = 0 \end{cases}
$$

and the Ansatz

$$
P_h f(x) = a \cdot 1 + b \cdot x = a + bx,
$$

from which a $(= -1/6)$ and b $(= 1)$ can be computed.

(b) We now divide I into the two subintervals $(0, \frac{1}{2})$ and $(\frac{1}{2}, 1)$. As in (a), we choose the "hat functions" as basis functions:

$$
\varphi_0 = \begin{cases} 1 - 2x, & x \in (0, \frac{1}{2}) \\ 0, & x \in (\frac{1}{2}, 1) \end{cases}
$$

$$
\varphi_1 = \begin{cases} 2x, & x \in (0, \frac{1}{2}) \\ 2 - 2x, & x \in (\frac{1}{2}, 1) \end{cases}
$$

$$
\varphi_2 = \begin{cases} 0, & x \in (0, \frac{1}{2}) \\ 2x - 1, & x \in (\frac{1}{2}, 1) \end{cases}
$$

Using the same technique as in (a), we obtain a 3×3 linear system of equations (since the number of nodes is 3 when the number of intervals is 2). The elements of the mass matrix are

$$\begin{aligned}
m_{00} &= \int_I \varphi_0 \varphi_0 \, dx \\
&= \int_0^{1/2} (1-2x)^2 \, dx \\
&= 1/6 \\
m_{10} &= \int_I \varphi_0 \varphi_1 \, dx \\
&= \int_0^{1/2} (1-2x) 2x \, dx \\
&= 1/12 \\
m_{20} &= \int_I \varphi_0 \varphi_2 \, dx \\
&= 0 \\
m_{11} &= \int_I \varphi_1 \varphi_1 \, dx \\
&= \int_0^{1/2} (2x)^2 \, dx + \int_{1/2}^1 (2-2x)^2 \, dx \\
&= 1/3 \\
m_{12} &= \int_I \varphi_2 \varphi_1 \, dx \\
&= \int_{1/2}^1 (2x-1)(2-2x) \, dx \\
&= 1/12 \\
m_{22} &= \int_I \varphi_2 \varphi_2 \, dx \\
&= \int_{1/2}^1 (2x-1)^2 \, dx \\
&= 1/6
\end{aligned}$$

Similarly, we get for the right hand side

$$\begin{aligned}
b_0 &= \int_I f \varphi_0 \, dx \\
&= \int_0^{1/2} x^2 (1-2x) \, dx \\
&= 1/96
\end{aligned}$$

$$
\begin{aligned}
b_1 &= \int_I f\varphi_1\,dx \\
&= \int_0^{1/2} x^2 2x\,dx + \int_{1/2}^1 x^2(2-2x)\,dx \\
&= 7/48 \\
b_2 &= \int_I f\varphi_2\,dx \\
&= \int_{1/2}^1 x^2(2x-1)\,dx \\
&= 17/96
\end{aligned}
$$

The system we have to solve is

$$
\begin{bmatrix} 1/6 & 1/12 & 0 \\ 1/12 & 1/3 & 1/12 \\ 0 & 1/12 & 1/6 \end{bmatrix}
\begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}
=
\begin{bmatrix} 1/96 \\ 7/48 \\ 17/96 \end{bmatrix}
$$

with the solution $c_0 = -1/24, c_1 = 5/24$ and $c_2 = 23/24$. Hence,

$$
\begin{aligned}
P_h f(x) &= c_0\,\varphi_0(x) + c_1\,\varphi_1(x) + c_2\,\varphi_2(x) \\
&= -1/24\,\varphi_0(x) + 5/24\,\varphi_1(x) + 23/24\,\varphi_2(x)
\end{aligned}
$$

$$
\left(
\begin{aligned}
&= \begin{cases} -1/24\cdot(1-2x) + 5/24\cdot 2x, & x \in (0,1/2) \\ 5/24\cdot(2-2x) + 23/24\cdot(2x-1), & x\in(1/2,1) \end{cases} \\
&= \begin{cases} -1/24 + x/2, & x\in(0,1/2) \\ -13/24 + 3x/2, & x\in(1/2,1) \end{cases}
\end{aligned}
\right)
$$

Remark. Cf. the Remark at the end of *Problem 4(a) (Week 1).*

Remark. Also in this case one might try the Ansatz

$$
P_h f(x) = \begin{cases} a + bx, & x\in(0,\frac{1}{2}) \\ c + dx, & x\in(\frac{1}{2},1) \end{cases}
$$

using $\{1,\, x\}$ as *local* basis functions on each subinterval. In addition to the orthogonality requirement (against three *global* basis functions, for instance $\{\varphi_i\}_{i=0}^2$) we will in this case need to enforce continuity at the point $x = 1/2$, and will therefore end up with 4 equations instead of 3, from which a $(= -1/24)$, b $(= 1/2)$, c $(= -13/24)$, d $(= 3/2)$, can be computed. This, however, is disadvantageous since we have to solve a linear system of four equations instead of three.

(c) See Figure 4 and Figure 5. □

Problem 4. Let $I = (0,1)$ and $0 = x_0 < x_1 < \cdots < x_N = 1$ be a partition of I into subintervals $I_j = (x_{j-1}, x_j)$ of length h_j.

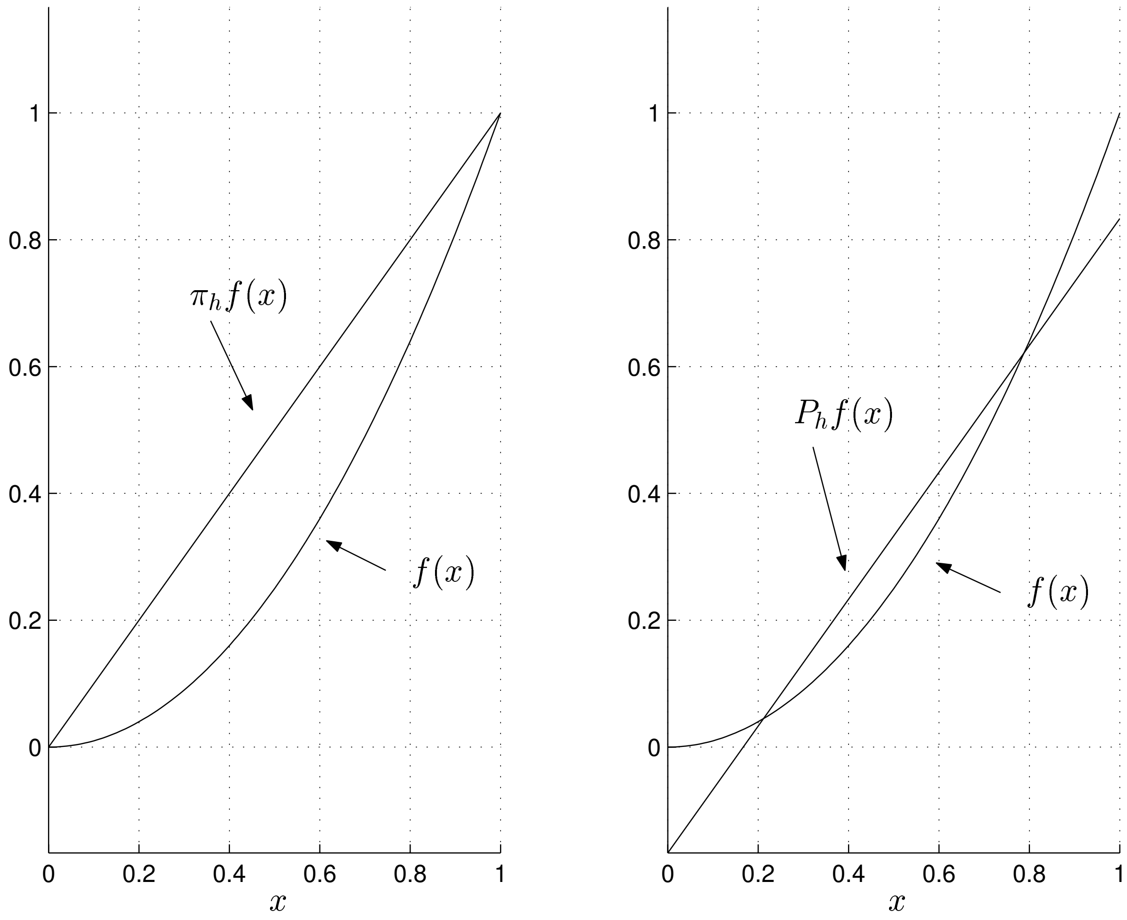

Figure 4: Problem 3(a) (Week 2). Plots of $f(x) = x^2$, $\pi_h f(x)$ and $P_h f(x)$.

(a) Assume $h_j = 1/N$ for all j. Calculate the mass matrix M.
(b) Calculate the mass matrix M in the general case.
Solution: The $(N+1) \times (N+1)$-matrix $M = (m_{ij})_{i,j=0}^N$ with elements

$$m_{ij} = \int_I \varphi_j \, \varphi_i \, dx, \tag{4}$$

where $\{\varphi_i\}_{i=0}^N \subset V_h$ are the nodal basis functions ("hat-functions"), is called the *mass matrix*.
(a) Look at the interval between say x_3 and x_4. On this interval there exist two non-zero basis functions φ_3 and φ_4. For $x \in [x_3, x_4]$ we have the following analytical expressions:

$$\varphi_3(x) = 1 - \frac{x - x_3}{h}, \quad \varphi_4(x) = \frac{x - x_3}{h}.$$

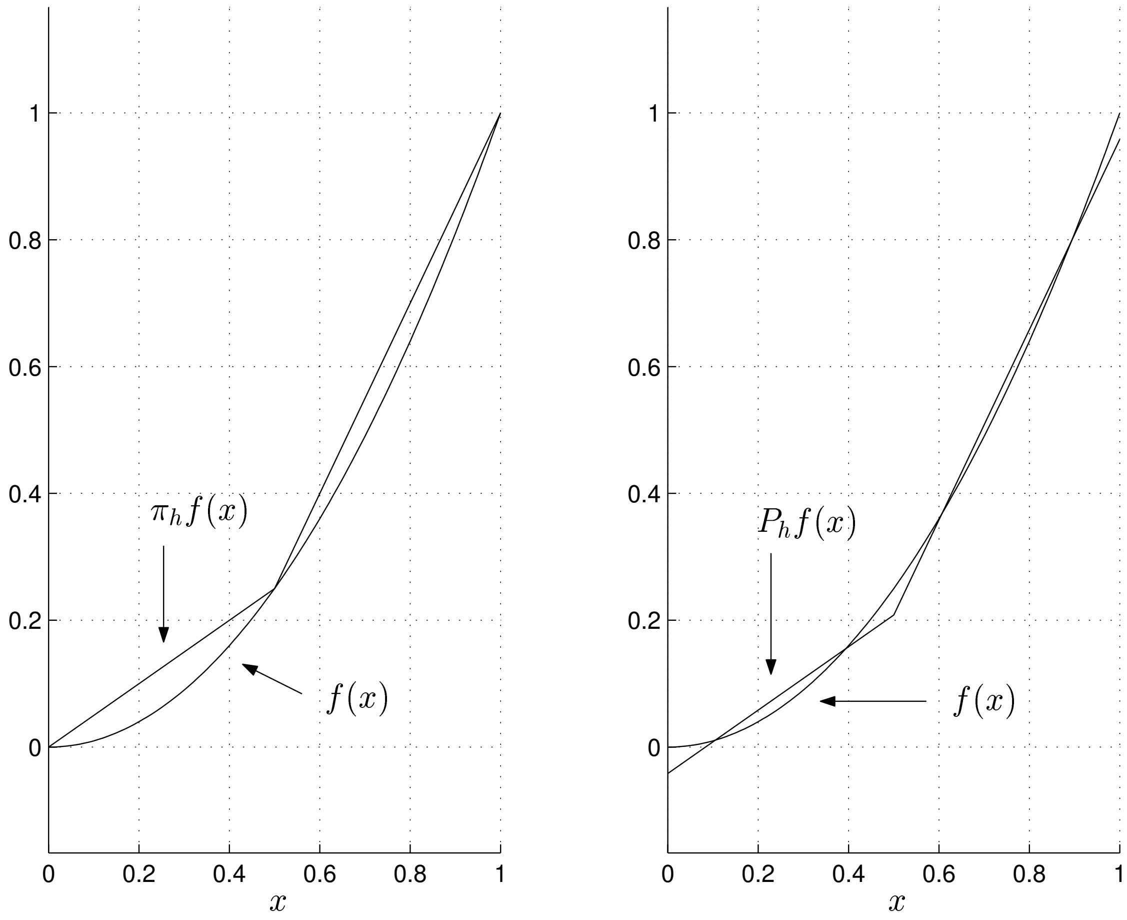

Figure 5: Problem 3(b) (Week 2). Plots of $f(x) = x^2$, $\pi_h f(x)$ and $P_h f(x)$.

This yields for the matrix elements m_{34} and m_{43}:

$$m_{34} = m_{43} = \int_0^1 \varphi_3(x)\, \varphi_4(x)\, dx = \int_{x_3}^{x_4} \varphi_3(x)\, \varphi_4(x)\, dx =$$

$$\int_{x_3}^{x_4} \left(1 - \frac{x - x_3}{h}\right) \cdot \frac{x - x_3}{h}\, dx = \{\text{Make a change of variables: } y = x - x_3\} =$$

$$\int_0^h \left(1 - \frac{y}{h}\right) \cdot \frac{y}{h}\, dy = \frac{h}{6},$$

since the integrand $\varphi_3(x)\, \varphi_4(x)$ is non-zero *only* for $x \in [x_3,\, x_4]$.

The interval $[x_3,\, x_4]$ also contributes to the matrix elements $m_{33} = \int_0^1 \varphi_3(x)\, \varphi_3(x)\, dx$ and $m_{44} = \int_0^1 \varphi_4(x)\, \varphi_4(x)\, dx$:

$$\frac{1}{2} \cdot m_{33} = \{\text{By symmetry }\} = \frac{1}{2} \cdot m_{44} = \int_{x_3}^{x_4} \varphi_4(x)\, \varphi_4(x)\, dx =$$

$$\int_{x_3}^{x_4} \frac{(x-x_3)^2}{h^2}\,dx = \{\text{Make a change of variables: } y = x - x_3\} = \int_0^h \frac{y^2}{h^2}\,dy = \frac{h}{3},$$

i.e., $m_{33} = m_{44} = 2h/3$, where the factor 2 compensates for the fact that φ_3 is non-zero on the interval $[x_2,\, x_4]$ and φ_4 is non-zero on the interval $[x_3,\, x_5]$. Thus, m_{33} and m_{44} get only half of their total value from the interval $[x_3,\, x_4]$.

Due to symmetry we may generalize to $m_{ii} = 2h/3,\ i = 1, \dots, N-1$, $m_{00} = m_{NN} = h/3$, $m_{i,i+1} = m_{i+1,i} = h/6,\ i = 0, \dots, N-1$, and $m_{ij} = 0$, otherwise. The exceptions for m_{00} and m_{NN} are due to the fact that the basis functions φ_0 and φ_N are just "half hats".

We summarize:

$$M = \begin{bmatrix} h/3 & h/6 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ h/6 & 2h/3 & h/6 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & h/6 & 2h/3 & h/6 & \dots & 0 & 0 & 0 & 0 \\ \\ \vdots & \vdots & & \ddots & \ddots & \ddots & & \vdots & \vdots \\ \\ 0 & 0 & 0 & 0 & \dots & h/6 & 2h/3 & h/6 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & h/6 & 2h/3 & h/6 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & h/6 & h/3 \end{bmatrix}$$

(b) We now look at the case where the interval $I = [0,\, 1]$ is non-uniformly partitioned. Consider once more the subinterval $[x_3,\, x_4]$. Simply replacing h by h_4 throughout in the computations in (a) gives $m_{34} = m_{43} = h_4/6$, and that the contribution from this subinterval to m_{33} and m_{44} is $h_4/3$. Adding the contributions from all subintervals now immediately generalizes the mass matrix computed in (a): $M =$

$$\begin{bmatrix} h_1/3 & h_1/6 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ h_1/6 & (h_1+h_2)/3 & h_2/6 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & h_2/6 & (h_2+h_3)/3 & h_3/6 & \dots & 0 & 0 & 0 & 0 \\ \\ \vdots & \vdots & & \ddots & \ddots & \ddots & & \vdots & \vdots \\ \\ 0 & 0 & 0 & 0 & \dots & h_{N-2}/6 & (h_{N-2}+h_{N-1})/3 & h_{N-1}/6 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & h_{N-1}/6 & (h_{N-1}+h_N)/3 & h_N/6 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & h_N/6 & h_N/3 \end{bmatrix}$$

$\square$

Problem 5. Recall that $(f, g) = \int_I fg\,dx$ and $\|f\|^2_{L^2(I)} = (f, f)$ are the L^2-scalar product and norm, respectively. Let $I = (0, \pi)$, $f = \sin x$, $g = \cos x$ for $x \in I$.

(a) Calculate (f, g).
(b) Calculate $\|f\|_{L^2(I)}$ and $\|g\|_{L^2(I)}$.
Solution:
(a) $(f, g) = \int_0^\pi \sin x \cos x \, dx = \frac{1}{2}[(\sin x)^2]_0^\pi = 0$.
(b) Recall the relations

$$\sin^2 x = \frac{1 - \cos 2x}{2}; \quad \cos^2 x = \frac{1 + \cos 2x}{2}.$$

Using these, we get:

$$||f||_{L^2(I)} = \sqrt{\int_0^\pi \sin^2 x \, dx} = \sqrt{\int_0^\pi \frac{1 - \cos 2x}{2} \, dx} = \sqrt{\frac{1}{2}\int_0^\pi \, dx - \frac{1}{2}\int_0^\pi \cos 2x \, dx}$$
$$= \sqrt{\frac{\pi}{2} - \frac{1}{4}[\sin 2x]_0^\pi} = \sqrt{\frac{\pi}{2}},$$

and, similarly,

$$||g||_{L^2(I)} = \sqrt{\int_0^\pi \cos^2 x \, dx} = \sqrt{\int_0^\pi \frac{1 + \cos 2x}{2} \, dx} = \sqrt{\frac{1}{2}\int_0^\pi \, dx + \frac{1}{2}\int_0^\pi \cos 2x \, dx} = \sqrt{\frac{\pi}{2}}.$$

$\square$

Problem 6. Show that $(f - P_h f, v) = 0, \quad \forall v \in V_h$, if and only if $(f - P_h f, \varphi_i) = 0, \quad i = 0, \ldots, N$; where $\{\varphi_i\}_{i=0}^N \subset V_h$ is the basis of hat-functions.
Solution:
$\Rightarrow$ Follows immediately since $\varphi_i \in V_h$ for $i = 0, \ldots, N$.
$\Leftarrow$ Assume that $(f - P_h f, \varphi_i) = 0$ for $i = 0, \ldots, N$. Since $v \in V_h$ and $\{\varphi_i\}_{i=0}^N$ is a *basis* for V_h, v can be written as $v = \sum_{i=0}^N \alpha_i \varphi_i$. This gives $(f - P_h f, v) = (f - P_h f, \sum_{i=0}^N \alpha_i \varphi_i) = \sum_{i=0}^N \alpha_i (f - P_h f, \varphi_i) = 0$ which proves the statement. $\square$

Problem 7. Let V be a linear subspace of $\mathbf{R}^n$ with basis $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m\}$ with $m < n$. Let $P\boldsymbol{x} \in V$ be the orthogonal projection of $\boldsymbol{x} \in \mathbf{R}^n$ onto the subspace V. Derive a linear system of equations that determines $P\boldsymbol{x}$. Note that your results are analogous to the L^2-projection when the usual scalar product in $\mathbf{R}^n$ is replaced by the scalar product in $L^2(I)$. Compare this method of computing the projection $P\boldsymbol{x}$ to the method used for computing the projection of a three dimensional vector onto a two dimensional subspace. What happens if the basis $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m\}$ is orthogonal?
Solution: Let $(\boldsymbol{u}, \boldsymbol{v})$ denote the usual scalar product in $\boldsymbol{R}^n$. Since $P\boldsymbol{x}$ is the orthogonal projection of $\boldsymbol{x} \in \boldsymbol{R}^n$ onto the subspace V of $\boldsymbol{R}^n$, we have

$$(\boldsymbol{x} - P\boldsymbol{x}, \boldsymbol{y}) = 0, \quad \text{for all } \boldsymbol{y} \in V.$$

Since $\{\boldsymbol{v}_1, ..., \boldsymbol{v}_m\}$ is a basis for V we may equivalently write (cf. *Problem 6 (Week 2)*)

$$(\boldsymbol{x} - P\boldsymbol{x}, \boldsymbol{v}_i) = 0, \quad i = 1, ..., m,$$

which leads to

$$(P\boldsymbol{x}, \boldsymbol{v}_i) = (\boldsymbol{x}, \boldsymbol{v}_i), \quad i = 1, ..., m.$$

But since $P\boldsymbol{x} \in V$ and $\{\boldsymbol{v}_1, ..., \boldsymbol{v}_m\}$ is a basis for V, $P\boldsymbol{x}$ can be written as a linear combination of elements in the basis, that is, $P\boldsymbol{x} = \sum_{j=1}^{m} \alpha_j \boldsymbol{v}_j$, $\alpha_j \in \boldsymbol{R}$. Inserting this above gives

$$(\sum_{j=1}^{m} \alpha_j \boldsymbol{v}_j, \boldsymbol{v}_i) = (\boldsymbol{x}, \boldsymbol{v}_i), \quad i = 1, ..., m,$$

or, using the linearity property of the scalar product,

$$\sum_{j=1}^{m} \alpha_j (\boldsymbol{v}_j, \boldsymbol{v}_i) = (\boldsymbol{x}, \boldsymbol{v}_i), \quad i = 1, ..., m,$$

which is a quadratic linear system of equations $A\alpha = b$, where $a_{ij} = (\boldsymbol{v}_j, \boldsymbol{v}_i)$ and $b_i = (\boldsymbol{x}, \boldsymbol{v}_i)$.

If the basis $\{\boldsymbol{v}_1, ..., \boldsymbol{v}_m\}$ is *orthogonal*, that is, $(\boldsymbol{v}_j, \boldsymbol{v}_i) = 0$ if $i \neq j$, the matrix A becomes *diagonal* and the equations simplify to

$$\alpha_i (\boldsymbol{v}_i, \boldsymbol{v}_i) = (\boldsymbol{x}, \boldsymbol{v}_i), \quad i = 1, ..., m,$$

which immediately gives

$$P\boldsymbol{x} = \sum_{j=1}^{m} \frac{(\boldsymbol{x}, \boldsymbol{v}_j)}{(\boldsymbol{v}_j, \boldsymbol{v}_j)} \boldsymbol{v}_j.$$

In the special case $n = 3$ and $m = 2$, which means computing the projection of a three dimensional vector $\boldsymbol{x}$ onto a two dimensional subspace, i.e., onto a *plane* through the origin, one usually computes $P\boldsymbol{x} = \boldsymbol{x} - \frac{(\boldsymbol{x},\boldsymbol{n})}{(\boldsymbol{n},\boldsymbol{n})} \boldsymbol{n}$, where $\boldsymbol{n}$ is a normal to the plane.

To compare the two methods, consider the case $\boldsymbol{n} = \boldsymbol{e}_3$, i.e., the plane $x_3 = 0$. Choosing the standard basis $\boldsymbol{v}_1 = \boldsymbol{e}_1$ and $\boldsymbol{v}_2 = \boldsymbol{e}_2$, we get $P\boldsymbol{x} = \boldsymbol{x} - (\boldsymbol{x}, \boldsymbol{e}_3)\, \boldsymbol{e}_3 = \boldsymbol{x} - x_3 \boldsymbol{e}_3 = x_1 \boldsymbol{e}_1 + x_2 \boldsymbol{e}_2 = (\boldsymbol{x}, \boldsymbol{e}_1)\, \boldsymbol{e}_1 + (\boldsymbol{x}, \boldsymbol{e}_2)\, \boldsymbol{e}_2$.

(Cf. *Applied Mathematics: B&S*, Part II, Section 21.17 *Projection of a point onto a plane.*) □

Week 3:

Problem 1. Let u be the solution to

$$-(au')' + cu = f \quad \text{in } (0,1), \tag{5}$$
$$u(0) = u(1) = 0, \tag{6}$$

where $a, c,$ and f are given functions.
(a) Show that u satisfies the variational equation

$$\int_0^1 (au'v' + cuv)\, dx = \int_0^1 fv\, dx, \tag{7}$$

for all sufficiently smooth v with $v(0) = v(1) = 0$.
(b) Introduce a partition of $(0,1)$ and the corresponding space of continuous piecewise linear functions V_{h0} which are zero for $x = 0$ and $x = 1$. Formulate a finite element method based on the variational equation in (a).
(c) Let $|||u||| = \left(\int_0^1 (au'u' + cuu)\, dx\right)^{1/2}$. Verify that $|||\cdot|||$ is a norm if $a(x) > 0$ and $c(x) \geq 0$ for all $x \in (0,1)$.
(d) Prove the a priori error estimate

$$|||u - U||| \leq |||u - v|||, \tag{8}$$

for all $v \in V_{h0}$.
(e) Assume that there are constants C_a and C_c such that $\|a\|_{L_\infty(0,1)} \leq C_a$ and $\|c\|_{L_\infty(0,1)} \leq C_c$, and that $\|u''\|_{L^2(0,1)}$ is bounded. Show that $|||u - U|||$ converges to zero as the meshsize tends to zero.
Solution:
(a) Multiply both sides of the differential equation by $v(x)$, such that $v(0) = v(1) = 0$, and integrate from $x = 0$ to $x = 1$ to get the following equality:

$$\int_0^1 (\,-(au')'v + cuv\,)\, dx = \int_0^1 fv\, dx.$$

Integrate by parts in the first term on the left-hand side, and use the fact that $v(0) = v(1) = 0$ to see that the boundary terms vanish:

$$\begin{aligned} -[au'v]_{x=0}^{x=1} + \int_0^1 (au'v' + cuv)\, dx &= \int_0^1 fv\, dx; \\ \int_0^1 (au'v' + cuv)\, dx &= \int_0^1 fv\, dx. \end{aligned}$$

(b) Let $0 = x_0 < x_1 < \cdots < x_N < x_{N+1} = 1$ be a partition of (0, 1) and let $\{\varphi_i\}_{i=1}^N$ be the "hat-functions" on this partition that are equal to one in an *internal* node. Define

$V_{h0} = \text{span}\{\varphi_1, \varphi_2, \ldots, \varphi_N\}$, i.e., V_{h0} is the vector space of continuous, piece-wise linear functions $v(x)$ that are zero at $x = 0$ and $x = 1$. The Finite Element Method now reads: Find $U \in V_{h0}$ such that

$$\int_0^1 (aU'v' + cUv)\,dx = \int_0^1 f v\,dx \quad \text{for all } v \in V_{h0}.$$

(c) To prove that $|||\cdot|||$ is a norm we must verify that:

(i) $|||u + v||| \leq |||u||| + |||v|||$ for all u and $v \in V_0$,

(ii) $|||\alpha u||| = |\alpha|\,|||u|||$ if $u \in V_0$ and $\alpha \in \boldsymbol{R}$,

(iii) $|||u||| = 0$ for $u \in V_0$ implies $u = \mathbf{0}$,

where V_0 denotes the vector space of functions that are zero at the boundary, and that are smooth enough for the integrals in the definition of $|||u|||$ to exist.

Since

$$|||u||| = (u,\, u)_E^{1/2},$$

where

$$(u,\, v)_E = \int_0^1 (\, a(x)u'(x)v'(x) + c(x)u(x)v(x)\,)\,dx,$$

is a *scalar product* between functions in V_0, property (i) follows from *the Cauchy-Schwarz inequality*:

$$\begin{aligned} |||u+v|||^2 &= (u+v,\, u+v)_E = (u,\, u)_E + 2(u,\, v)_E + (v,\, v)_E \\ &\leq |||u|||^2 + 2|||u|||\cdot|||v||| + |||v|||^2 = (|||u||| + |||v|||)^2. \end{aligned}$$

Property (ii) follows since

$$\int_0^1 (\, a(x)(\alpha u'(x))^2 + c(x)(\alpha u(x))^2\,)\,dx = \alpha^2 \int_0^1 (\, a(x)u'(x)^2 + c(x)u(x)^2\,)\,dx.$$

To prove property (iii) we notice that $a(x)u'(x)^2 \geq 0$ and $c(x)u(x)^2 \geq 0$. This means that $\int_0^1 a(x)u'(x)^2\,dx \geq 0$ and $\int_0^1 c(x)u(x)^2\,dx \geq 0$. If $0 = |||u|||^2 = \int_0^1 a(x)u'(x)^2\,dx + \int_0^1 c(x)u(x)^2\,dx$, both these integrals must therefore be equal to zero. Since $a(x) > 0$ this implies $u'(x) \equiv 0$, which means that $u(x) \equiv K$ where K is a constant. But since $u(0) = u(1) = 0$ we must have $K = 0$.

Remark. If $c(x) > 0$ is (also) *strictly positive* then $\int_0^1 c(x)u(x)^2\,dx = 0$ immediately implies that $u(x) \equiv 0$ and we don't need to use the boundary conditions.

(d) Observe that, by using the definition of $(u,\, v)_E$ in (c), the variational equation in (a) can be written

$$(u,\, v)_E = \int_0^1 f v\, dx \quad \text{for all } v \in V_0,$$

and the Finite Element Method in (b) can be written

$$(U,\, v)_E = \int_0^1 f v\, dx \quad \text{for all } v \in V_{h0}.$$

Since $V_{h0} \subset V_0$ we get by subtracting:

$$(u - U,\, v)_E = 0 \quad \text{for all } v \in V_{h0}.$$

The last equation expresses *the Galerkin orthogonality.* This shows that the Finite Element approximation $U(x)$ of $u(x)$ is *the orthogonal projection* of u onto V_{h0} with respect to the scalar product $(\cdot,\, \cdot)_E$. This orthogonality, and the Cauchy-Schwarz inequality, implies that for an *arbitrary* function $v(x) \in V_{h0}$:

$$|||u - U|||^2 = (u - U,\, u - U)_E = (u - U,\, u - U + (U - v))_E$$

$$= (u - U,\, u - v)_E \leq |||u - U||| \cdot |||u - v|||,$$

since $U - v \in V_{h0}$. Dividing both sides by $|||u - U|||$ now completes the proof.

Remark. Observe the complete analogy between this proof and the corresponding proof for the L^2-projection.

(e) Assume for simplicity that the partition is uniform, i.e., that the mesh function $h(x) \equiv h$ is a constant function. Choosing v in (d) to be the nodal interpolant $\pi_h u(x) \in V_{h0}$ of u, we get:

$$|||u - U|||^2 \leq |||u - \pi_h u|||^2$$

$$= \int_0^1 (\, a(x)(u - \pi_h u)'(x)^2 + c(x)(u - \pi_h u)(x)^2 \,)\, dx$$

$$\leq C_a \int_0^1 (u - \pi_h u)'(x)^2\, dx + C_c \int_0^1 (u - \pi_h u)(x)^2\, dx$$

$$= C_a ||(u - \pi_h u)'||^2_{L^2(0,1)} + C_c ||u - \pi_h u||^2_{L^2(0,1)}$$

$$\leq C_a C_i^2 ||h u''||^2_{L^2(0,1)} + C_c C_i^2 ||h^2 u''||^2_{L^2(0,1)}$$

$$= C_a C_i^2 h^2 \|u''\|^2_{L^2(0,1)} + C_c C_i^2 h^4 \|u''\|^2_{L^2(0,1)},$$

which tends to zero as h tends to zero. (C_i denotes interpolation constants.) □

Problem 2. Let u be the solution to

$$-u''(x) = 1 \quad \text{in } (0,1), \tag{9}$$

$$u(0) = u(1) = 0. \tag{10}$$

(a) Solve the problem analytically.
(b) Let $I = (0, 1)$ be divided into a uniform mesh with $h = 1/N$. Calculate (by hand) the finite element approximation U for $N = 2, 3$.
(c) Plot your solutions in a figure. Compare your results.
Solution:
(a) Integrating the differential equation twice gives:

$$u''(x) = -1 \quad \Rightarrow \quad u'(x) = -x + C_1 \quad \Rightarrow \quad u(x) = -x^2/2 + C_1 x + C_2.$$

The boundary condition $u(0) = 0$ then gives $C_2 = 0$, and $u(1) = 0$ gives $-1/2+C_1+C_2 = 0$, i.e., $C_1 = 1/2$; $\quad C_2 = 0$. Therefore:

$$u(x) = -\frac{x^2}{2} + \frac{x}{2} = \frac{x(1-x)}{2}.$$

(b) The finite element approximation $U(x) = \sum_{j=1}^{M} \xi_j \varphi_j(x)$ can be computed by solving the linear system of equations (see *Applied Mathematics: B&S*, Part D, equation 54.4, with $a = 1$):

$$\sum_{j=1}^{M} \xi_j \int_0^1 \varphi_j' \varphi_i' \, dx = \int_0^1 f \varphi_i \, dx \quad i = 1, \ldots, M,$$

which determines the unknown coefficients $\xi_1, \ldots, \xi_M$. Here M is the number of *internal* nodes, since we have *homogeneous Dirichlet boundary conditions.*

If the number of subintervals is $N = 2$, then there is only one internal node, $M = 1$, and the equation above simplifies to:

$$\xi_1 \int_0^1 \varphi_1' \varphi_1' \, dx = \int_0^1 f \varphi_1 \, dx.$$

Since $f(x) = 1$, $\varphi_1' = 2$ on $[0, \frac{1}{2}]$ and $\varphi_1' = -2$ on $[\frac{1}{2}, 1]$, we get

$$\xi_1 \left(\int_0^{0.5} 2^2 \, dx + \int_{0.5}^1 (-2)^2 \, dx \right) = 4\xi_1 = \int_0^1 \varphi_1 \, dx = \frac{1}{2},$$

which gives that $\xi_1 = \frac{1}{8}$. That is: $U(x) = \frac{1}{8} \varphi_1(x)$.

Remark. The integral $\int_0^1 \varphi_1\,dx$ is geometrically the *area* under φ_1, i.e., the area of a triangle.

If the number of subintervals is $N = 3$, then there are two internal nodes, $M = 2$, and we get the following linear system of equations:

$$\xi_1 \int_0^1 \varphi_1' \varphi_1'\,dx + \xi_2 \int_0^1 \varphi_2' \varphi_1'\,dx = \int_0^1 f \varphi_1\,dx,$$
$$\xi_1 \int_0^1 \varphi_1' \varphi_2'\,dx + \xi_2 \int_0^1 \varphi_2' \varphi_2'\,dx = \int_0^1 f \varphi_2\,dx.$$

Since $f(x) = 1$ and

$$\varphi_1'(x) = \begin{cases} 0, & x \notin [0, \frac{2}{3}], \\ 3, & x \in [0, \frac{1}{3}], \\ -3, & x \in [\frac{1}{3}, \frac{2}{3}], \end{cases} \qquad \varphi_2'(x) = \begin{cases} 0, & x \notin [\frac{1}{3}, 1], \\ 3, & x \in [\frac{1}{3}, \frac{2}{3}], \\ -3, & x \in [\frac{2}{3}, 1], \end{cases}$$

we get:

$$\xi_1 \left(\int_0^{\frac{1}{3}} 3^2\,dx + \int_{\frac{1}{3}}^{\frac{2}{3}} (-3)^2\,dx \right) + \xi_2 \int_{\frac{1}{3}}^{\frac{2}{3}} 3(-3)\,dx = 6\xi_1 - 3\xi_2 = \int_0^1 \varphi_1\,dx = \frac{1}{3},$$
$$\xi_1 \int_{\frac{1}{3}}^{\frac{2}{3}} (-3)3\,dx + \xi_2 \left(\int_{\frac{1}{3}}^{\frac{2}{3}} 3^2\,dx + \int_{\frac{2}{3}}^{1} (-3)^2\,dx \right) = -3\xi_1 + 6\xi_2 = \int_0^1 \varphi_2\,dx = \frac{1}{3},$$

with solution $\xi_1 = \xi_2 = \frac{1}{9}$. That is: $U(x) = \frac{1}{9}\,\varphi_1(x) + \frac{1}{9}\,\varphi_2(x)$.
(c) See Figure 6. □

Problem 3*.
(a) Show that the finite element approximations U that you have computed in *Problem 2 (Week 3)* actually are exactly equal to u *at the nodes*, by simply evaluating u and U at the nodes.
(b) Prove this result. *Hint:* Show that the error $e = u - U$ can be written

$$e(z) = \int_0^1 g_z'(x) e'(x)\,dx, \quad 0 \le z \le 1,$$

where

$$g_z(x) = \begin{cases} (1-z)x, & 0 \le x \le z, \\ z(1-x), & z \le x \le 1, \end{cases}$$

and then use the fact the $g_{x_j} \in V_{h0}$.
(c) Does the result in (b) extend to variable $a = a(x)$?
Solution:
(a) From *Problem 2 (Week 3)* with $N = 2$ we get

$$u(1/2) = \frac{1}{2}(1 - \frac{1}{2})/2 = 1/8,$$

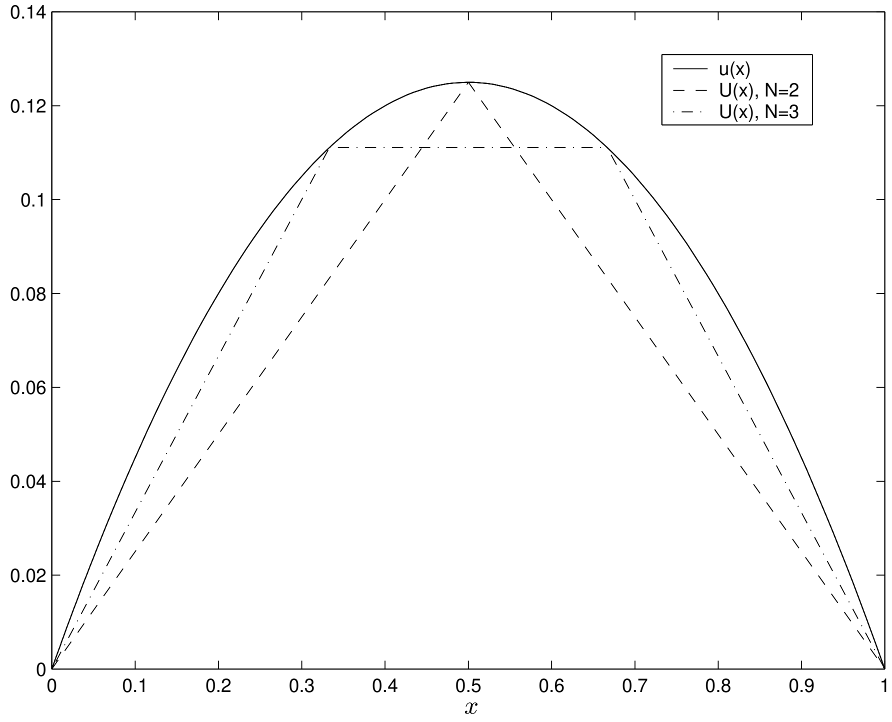

Figure 6: Problem 2 (Week 3). Plots of $u(x)$ and $U(x)$ for $N = 2, 3$.

and

$$U(1/2) = \frac{1}{8}\,\varphi_1(1/2) = 1/8.$$

Hence, $u(1/2) = U(1/2)$.

Using $N = 3$ we have for the first inner node

$$u(1/3) = \frac{1}{3}(1 - \frac{1}{3})/2 = 1/9,$$

and

$$U(1/3) = \frac{1}{9}\,\varphi_1(1/3) + \frac{1}{9}\,\varphi_2(1/3) = \frac{1}{9} \cdot 1 + 0 = \frac{1}{9}.$$

For the second inner node:

$$u(2/3) = \frac{2}{3}(1 - \frac{2}{3})/2 = 1/9,$$

and

$$U(2/3) = \frac{1}{9}\,\varphi_1(2/3) + \frac{1}{9}\,\varphi_2(2/3) = 0 + \frac{1}{9} \cdot 1 = \frac{1}{9}.$$

Hence, $u(1/3) = U(1/3)$ and $u(2/3) = U(2/3)$.

(b) To check the given formula for $e(z)$ we must compute the integral. Before we can do that, we must calculate the derivative of $g_z(x)$:

$$g_z'(x) = \frac{dg_z(x)}{dx} = \begin{cases} 1 - z, & 0 \le x < z, \\ -z, & z < x \le 1. \end{cases}$$

Thus, we have:

$$\begin{aligned} \int_0^1 g_z'(x)e'(x)\,dx &= \int_0^z (1-z)e'(x)\,dx + \int_z^1 -ze'(x)\,dx \\ &= (1-z)(e(z) - e(0)) - z(e(1) - e(z)) \\ &= e(z) - \underbrace{e(0) + ze(0) - ze(1)}_{=0} \\ &= e(z), \end{aligned}$$

since the error $e = u - U$ is equal to zero at the boundary points $x = 0$ and $x = 1$. This follows from the boundary conditions, $u(0) = U(0) = 0$ and $u(1) = U(1) = 0$.

To show that the error is zero also at all internal nodal points x_j, we only need to show that $g_{x_j} \in V_{h0}$. The result then follows from *the Galerkin orthogonality* (cf. *Problem 1(d) (Week 3)* with $a = 1$ and $c = 0$), $\int_0^1 e'v'\,dx = (e,\, v)_E = 0$ for all $v \in V_{h0}$, by taking $v = g_{x_j}$. But from Figure 7 we see that g_{x_j} can be written as

$$g_{x_j}(x) = \sum c_i\,\varphi_i(x)$$

with weights $c_i = g_{x_j}(x_i)$. Hence, $g_{x_j} \in V_{h0}$. Also note that $g_z(x) \notin V_{h0}$ if $z \neq x_j$, which can be seen from Figure 8.
(c) No. As a counter-example, consider the case $a(x) = 1 + x$:

$$\begin{aligned} -((1+x)u')' = 1, \quad 0 < x < 1, \\ u(0) = u(1) = 0. \end{aligned}$$

The solution is $u(x) = \frac{\log(1+x)}{\log(2)} - x$. Computing the Finite Element approximation $U(x)$ for $N = 2$ in the same way as in *Problem 2(b) (Week 3)* gives $U(x) = \frac{1}{12}\varphi_1(x)$. We thus have that $U(1/2) = \frac{1}{12} \neq \frac{\log(3/2)}{\log(2)} - \frac{1}{2} = u(1/2)$. □

Problem 4. Consider the system of ODE:

$$M\dot{\xi}(t) + A\xi(t) = b \quad \text{in } (0,\, T), \tag{11}$$
$$\xi(0) = \xi^0. \tag{12}$$

Assume that

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 4 & 14 \\ 4 & 8 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \xi^0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{13}$$

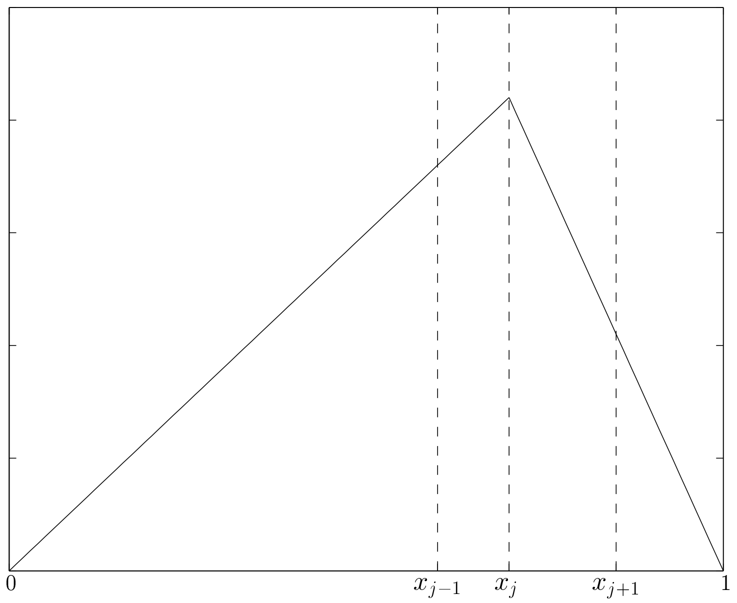

Figure 7: Problem 3 (Week 3). $g_z(x)$ when $z = x_j$.

Make a uniform partition of the time interval $(0, 1)$ into two sub-intervals and compute an approximation of $\xi(1)$ with the *backward Euler* method.
Solution: We divide the time interval: $0 = t_0 < t_1 < t_2 = 1$, with $t_1 = 0.5$, i.e., into two subintervals with length $\triangle t = 0.5$. The Euler backward method approximates the time derivative with a difference quotient in the following manner:

$$\begin{aligned} M\frac{\xi^n - \xi^{n-1}}{\triangle t} + A\xi^n = b, \quad n = 1,\, 2, \\ \xi^0 = \xi(0). \end{aligned}$$

So to compute $\xi^2 \approx \xi(t_2)$ we have to solve, in order, the equations:

$$M\frac{\xi^1 - \xi^0}{\triangle t} + A\xi^1 = b,$$

$$M\frac{\xi^2 - \xi^1}{\triangle t} + A\xi^2 = b.$$

Rearrangement of the first of these equations yields:

$$M\xi^1 + \triangle t\, A\xi^1 = M\xi^0 + \triangle t\, b;$$

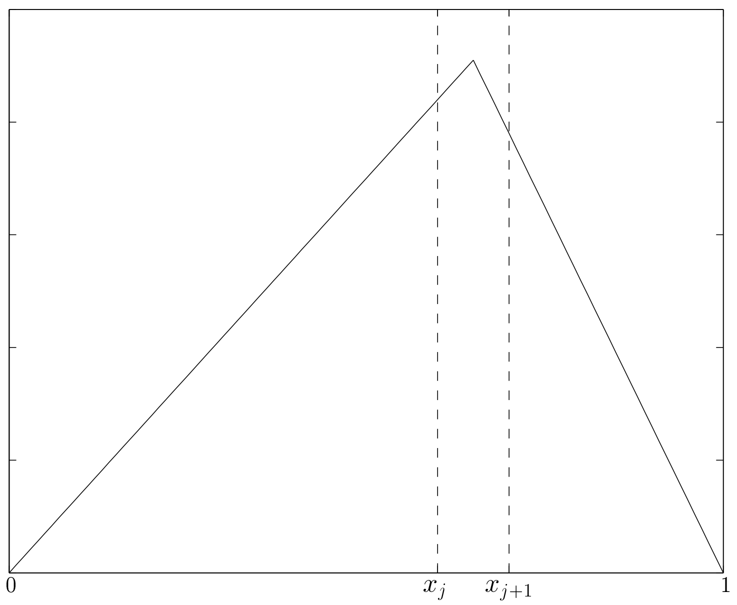

Figure 8: Problem 3 (Week 3). $g_z(x)$ when $z \neq x_j$.

$$\begin{aligned}
(M + \triangle t\, A)\xi^1 =& M\xi^0 + \triangle t\, b;\\
\left(\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \tfrac{1}{2}\begin{bmatrix}4 & 14\\ 4 & 8\end{bmatrix}\right)\xi^1 =& \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix} + \tfrac{1}{2}\begin{bmatrix}0\\ 0\end{bmatrix};\\
\begin{bmatrix}3 & 7\\ 2 & 5\end{bmatrix}\xi^1 =& \begin{bmatrix}1\\ 1\end{bmatrix};\\
\xi^1 =& \begin{bmatrix}-2\\ 1\end{bmatrix},
\end{aligned}$$

where the linear system of equations is solved by *Gaussian elimination*. Similarly, we get for the second equation:

$$\begin{aligned}
M\xi^2 + \triangle t\, A\xi^2 =& M\xi^1 + \triangle t\, b;\\
(M + \triangle t\, A)\xi^2 =& M\xi^1 + \triangle t\, b;\\
\left(\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \tfrac{1}{2}\begin{bmatrix}4 & 14\\ 4 & 8\end{bmatrix}\right)\xi^2 =& \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}-2\\ 1\end{bmatrix} + \tfrac{1}{2}\begin{bmatrix}0\\ 0\end{bmatrix};\\
\begin{bmatrix}3 & 7\\ 2 & 5\end{bmatrix}\xi^2 =& \begin{bmatrix}-2\\ 1\end{bmatrix};\\
\xi^2 =& \begin{bmatrix}-17\\ 7\end{bmatrix}.
\end{aligned}$$

The vector $\xi^2 = \begin{bmatrix}-17\\ 7\end{bmatrix}$ is thus an approximation of the solution $\xi(t)$ at time $t = 1$ (and

$\xi^1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ at time $t = 0.5$). $\square$

Problem 5. Show that, for the time dependent reaction-diffusion problem with Robin boundary conditions,

$$\begin{aligned}
\dot{u} - (au')' + cu &= f(x,t), \quad x_{\min} < x < x_{\max}, \quad 0 < t < T, \\
a(x_{\min})u'(x_{\min},t) &= \gamma(x_{\min})(u(x_{\min},t) - g_D(x_{\min})) + g_N(x_{\min}), \quad 0 < t < T, \\
-a(x_{\max})u'(x_{\max},t) &= \gamma(x_{\max})(u(x_{\max},t) - g_D(x_{\max})) + g_N(x_{\max}), \quad 0 < t < T, \\
u(x,0) &= u_0(x), \quad x_{\min} < x < x_{\max},
\end{aligned}$$

semi-discretization in space leads to the following system of ODE:

$$M\,\dot{\xi}(t) \,+\, (A \,+\, M_c \,+\, R)\,\xi(t) \,=\, b(t) + rv, \quad 0 < t < T.$$

Solution: *Hint:* To derive the variational formulation, first multiply both sides of the differential equation by a function $v = v(x)$. Then integrate both sides from $x = x_{\min}$ to $x = x_{\max}$. Integrate by parts in "the diffusive term" $\int_{x_{\min}}^{x_{\max}} (\,-(au')'v\,)\,dx$. Finally use the boundary conditions to *replace* au' in the boundary terms at $x = x_{\min}, x_{\max}$. This gives *the variational formulation*:

Find $u(x,t)$ such that for every fixed t: $u(x,t) \in V$, and

$$\int_{x_{\min}}^{x_{\max}} \dot{u}v\,dx \,+\, \gamma uv|_{x=x_{\max}} \,+\, \gamma uv|_{x=x_{\min}} \,+\, \int_{x_{\min}}^{x_{\max}} au'v'\,dx \,+\, \int_{x_{\min}}^{x_{\max}} cuv\,dx \,=$$

$$(\gamma g_D - g_N)v|_{x=x_{\max}} \,+\, (\gamma g_D - g_N)v|_{x=x_{\min}} \,+\, \int_{x_{\min}}^{x_{\max}} fv\,dx, \quad 0 < t < T, \quad \text{for all } v \in V,$$

where V is the vector space of functions $v = v(x)$ that are smooth enough for the integrals in the variational formulation to exist.

The corresponding *Finite Element Method* reads:

Find $U(x,t)$ such that for every fixed t: $U(x,t) \in V_h$, and

$$\int_{x_1}^{x_N} \dot{U}v\,dx \,+\, \gamma Uv|_{x=x_N} \,+\, \gamma Uv|_{x=x_1} \,+\, \int_{x_1}^{x_N} aU'v'\,dx \,+\, \int_{x_1}^{x_N} cUv\,dx \,=$$

$$(\gamma g_D - g_N)v|_{x=x_N} \,+\, (\gamma g_D - g_N)v|_{x=x_1} \,+\, \int_{x_1}^{x_N} fv\,dx, \quad 0 < t < T, \quad \text{for all } v \in V_h,$$

where V_h is the vector space of functions $v = v(x)$ that are continuous and piecewise linear on a partition $x_{\min} = x_1 < x_2 < \ldots < x_N = x_{\max}$ of $[x_{\min},\, x_{\max}]$.

Finally, insert the *Ansatz*

$$U(x,t) = \sum_{j=1}^{N} \xi_j(t)\varphi_j(x),$$

into the Finite Element formulation and choose $v = \varphi_i$ for $i = 1, \ldots, N$.

Week 5:

Problem 1. Consider the triangulation of the unit square $\Omega = [0,\ 1] \times [0,\ 1]$ into 8 triangles drawn in Figure 9.

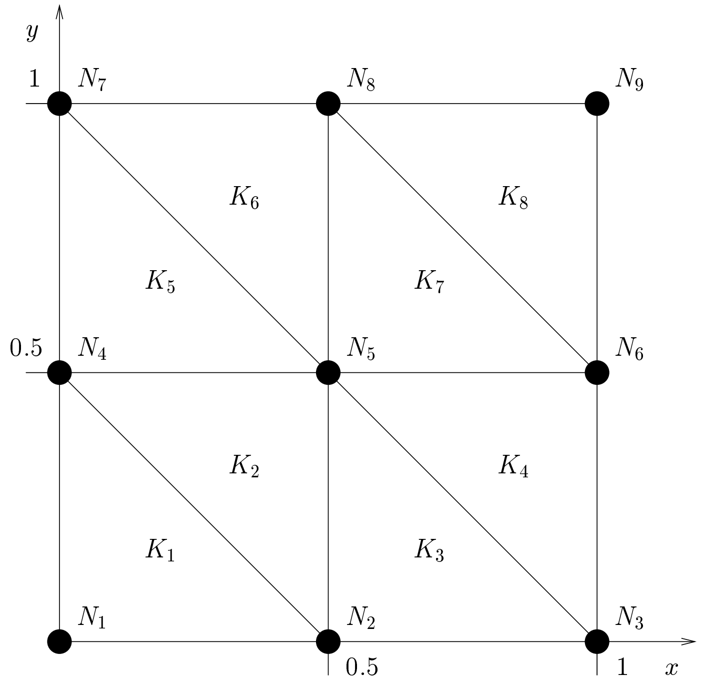

Figure 9: Problem 1 (Week 5). The triangulation of Ω.

(a) Compute the length of the largest side h_{K_j}, and the smallest angle τ_{K_j} of the triangles.
(b) Determine the *point matrix* `p` that describes this triangulation in Matlab. *Hint:* Since node 1 is located at the origin, the first column in `p` is `[0; 0]`.
(c) Determine the *triangle matrix* `t` that describes this triangulation in Matlab. *Hint:* Since triangle 1 has corners in node number 1, 2 and 4, the first column in `t` can e.g. be `[1; 2; 4]`. It is not important which node comes first, but they must be listed in a *counter-clockwise* order.
(d) Verify your results by creating `p` and `t` in Matlab:

```
>> p(:, 1) = [0; 0]
>> p(:, 2) = ...
...
>> p(:, 9) = ...
>> t(:, 1) = [1; 2; 4]
>> t(:, 2) = ...
...
>> t(:, 8) = ...
```

and plot the triangulation by the Matlab-command:

```
>> pdemesh(p, [], t)
```

Solution:
(a) Pythagoras' theorem and simple trigonometry gives that $h_{K_j} = 1/\sqrt{2}$ and $\tau_{K_j} = \pi/4$ for all triangles K_j.
(b)

$$p = \begin{bmatrix} 0 & 0.5 & 1 & 0 & 0.5 & 1 & 0 & 0.5 & 1 \\ 0 & 0 & 0 & 0.5 & 0.5 & 0.5 & 1 & 1 & 1 \end{bmatrix}$$

(c) For example:

$$t = \begin{bmatrix} 1 & 2 & 2 & 3 & 4 & 5 & 5 & 6 \\ 2 & 5 & 3 & 6 & 5 & 8 & 6 & 9 \\ 4 & 4 & 5 & 5 & 7 & 7 & 8 & 8 \end{bmatrix}$$

(d) - □

Problem 2. Consider the same triangulation as in *Problem 1 (Week 5)*.
(a) The continuous piecewise linear function $\varphi_2(x, y)$ is defined by:

$$\varphi_2(N_2) = 1; \quad \varphi_2(N_j) = 0 \text{ for } j \neq 2.$$

Compute the analytical expression for φ_2. *Hint:* The analytical expressions on K_1, K_2 and K_3 may be determined by solving linear systems of equations as you have seen in the lecture. On the other triangles, $\varphi_2 \equiv 0$. Why?
(b) Plot φ_2 in Matlab by giving the command:

```
>> pdesurf(p, t, [0; 1; 0; 0; 0; 0; 0; 0; 0])
```

or

```
>> pdemesh(p, [], t, [0; 1; 0; 0; 0; 0; 0; 0; 0])
```

Try both! The argument `[0; 1; 0; 0; 0; 0; 0; 0; 0]` is a *column vector* containing the *nodal values* of φ_2. Try also to plot some other "tent functions" φ_j!
(c) Since an arbitrary continuous piecewise linear function v can be written as a linear combination of "tent functions":

$$v(x, y) = v(N_1)\,\varphi_1(x, y) + \ldots + v(N_9)\,\varphi_9(x, y)$$

the "tent functions" $\{\varphi_i\}_{i=1}^{9}$ form a *basis* for the vector space V_h of continuous piecewise linear functions on the triangulation in Figure 9. What is the *dimension* of V_h?
(d) Try plotting some different functions in V_h using the Matlab commands `pdesurf` and `pdemesh`. *Hint:* Cf. how you plotted φ_2.

Solution:
(a) The analytical expression for φ_2 is different on each triangle. Since φ_2 is equal to one at node N_2 and zero at all other nodes, it is only non-zero on triangles K_1, K_2 and K_3, and therefore $\varphi_2(x,\, y) = 0$ on $K_4 \cup K_5 \cup K_6 \cup K_7 \cup K_8$.

On each triangle φ_2 is a linear function $\varphi_2(x,\, y) = c_0 + c_1 x + c_2 y$, where c_0, c_1, and c_2 are constants to be determined for the three triangles K_1, K_2 and K_3. These constants can be computed by solving the linear system of equations (see *Applied Mathematics: B&S*, Part D, page 815):

$$\begin{pmatrix} 1 & a_1^1 & a_2^1 \\ 1 & a_1^2 & a_2^2 \\ 1 & a_1^3 & a_2^3 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \varphi_2(a_1^1, a_2^1) \\ \varphi_2(a_1^2, a_2^2) \\ \varphi_2(a_1^3, a_2^3) \end{pmatrix}$$

where (a_1^1, a_2^1), (a_1^2, a_2^2) and (a_1^3, a_2^3) are the node coordinates of the triangle. On triangle K_1, with nodes N_1, N_2 and N_4 (in that order), we get:

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0.5 & 0 \\ 1 & 0 & 0.5 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

which has the solution $c_0 = 0$, $c_1 = 2$ and $c_2 = 0$. That is: on K_1, $\varphi_2(x,\, y) = c_0 + c_1 x + c_2 y = 2x$.

Similarly, on triangle K_2, with nodes N_2, N_5 and N_4 (in that order), we get:

$$\begin{pmatrix} 1 & 0.5 & 0 \\ 1 & 0.5 & 0.5 \\ 1 & 0 & 0.5 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

which has the solution $c_0 = 1$, $c_1 = 0$ and $c_2 = -2$. That is: on K_2, $\varphi_2(x,\, y) = 1 - 2y$.

Finally, on triangle K_3, with nodes N_2, N_3 and N_5 (in that order), we get:

$$\begin{pmatrix} 1 & 0.5 & 0 \\ 1 & 1 & 0 \\ 1 & 0.5 & 0.5 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

which has the solution $c_0 = 2$, $c_1 = -2$ and $c_2 = -2$. That is: on K_3, $\varphi_2(x,\, y) = c_0 + c_1 x + c_2 y = 2 - 2(x + y)$.
(b) -
(c) The dimension of V_h is 9, since there are 9 nodes and therefore 9 basis functions.
(d) To plot $v(x,\, y) = 2\,\varphi_1(x,\, y) + 3\,\varphi_8(x,\, y)$:

```
>> pdesurf(p, t, [2; 0; 0; 0; 0; 0; 0; 3; 0])
```

or

```
>> pdemesh(p, [], t, [2; 0; 0; 0; 0; 0; 0; 3; 0])
```

Comment: `[]` is an empty matrix. We don't need to use this argument but we still have to pass something to the function `pdemesh`, which expects an argument (actually the "edge matrix" `e`) between `p` and `t`. See `>> help pdemesh` □

Week 6:

Problem 1. Calculate $\|f\|_{L_\infty(\Omega)}$ where $\Omega = [0,1] \times [0,1]$ and
(a) $f(x_1, x_2) = x_2^2\,(x_1 - 2/3)^3$. Hint: To compute $\max_{(x_1,x_2)\in\Omega} |f(x_1, x_2)|$, maximize the absolute value of each factor of f separately.
(b) $f(x_1, x_2) = 11/36 - x_1^2 + x_1 - x_2^2 + 8x_2/3$. Hint: Compute both $\max_{(x_1,x_2)\in\Omega} f(x_1, x_2)$ and $\min_{(x_1,x_2)\in\Omega} f(x_1, x_2)$.
Solution:
(a) Since $\|f\|_{L_\infty(\Omega)} = \max_{(x_1,x_2)\in\Omega} |f(x_1, x_2)|$ we want to find the maximum of the absolute value $|f(x_1, x_2)|$ of $f(x_1, x_2)$. From the hint we start by maximising the x_2-dependent factor over the interval $[0, 1]$: The result is trivially 1 (for $x_2 = 1$). The maximum of the absolute value of the x_1-dependent factor is $8/27$ for $x_1 = 0$. This means that $\|f\|_{L_\infty(\Omega)} = 8/27$.
(b) We complete the squares to get:

$$f(x_1, x_2) = 11/36 - x_1^2 + x_1 - x_2^2 + 8x_2/3 = 7/3 - (x_1 - 1/2)^2 - (x_2 - 4/3)^2$$

We can now determine the maximum by minimising the two negative terms over Ω: Maximum of f thus occurs for $x_1 = 1/2$ and $x_2 = 1$ which gives us that $\max_{(x_1,x_2)\in\Omega} f(x_1, x_2) = 7/3 - 1/9 = 20/9$. In the same way minimum occurs when the last two terms are maximal, i.e., for $x_1 = 0$ or $x_1 = 1$ and $x_2 = 0$. Hence $\min_{(x_1,x_2)\in\Omega} f(x_1, x_2) = 7/3 - 1/4 - 16/9 = 11/36$. Since the minimum is positive, $f(x_1, x_2) = |f(x_1, x_2)|$ in Ω, and we conclude that $\|f\|_{L_\infty(\Omega)} = \max_{(x_1,x_2)\in\Omega} f(x_1, x_2) = 20/9$. $\square$

Problem 2. Calculate $\|f\|_{L^2(\Omega)}$ where $\Omega = [0,1] \times [0,1]$ and
(a) $f(x_1, x_2) = x_1\, x_2^2$.
(b) $f(x_1, x_2) = \sin(n\pi x_1)\sin(m\pi x_2)$ with n and m arbitrary integers.
Hint: $\sin^2 u = \frac{1-\cos(2u)}{2}$
Solution: The $L^2(\Omega)$-norm of f is defined by: $\|f\|_{L^2(\Omega)} = (\iint_\Omega f(x_1, x_2)^2\, dx_1\, dx_2)^{\frac{1}{2}}$.
(a)

$$\|f\|_{L^2(\Omega)}^2 = \int_0^1\int_0^1 x_1^2 x_2^4\, dx_1\, dx_2 = \int_0^1 x_1^2\, dx_1 \int_0^1 x_2^4\, dx_2 = [x_1^3/3]_0^1 \cdot [x_2^5/5]_0^1 = \frac{1}{15}$$

so $\|f\|_{L^2(\Omega)} = \frac{1}{\sqrt{15}}$.
(b) If n and/or m is equal to zero then f is identically equal to zero implying that $\|f\|_{L^2(\Omega)} = 0$. Otherwise we get:

$$\begin{aligned}
\|f\|_{L^2(\Omega)}^2 &= \int_0^1\int_0^1 \sin^2(n\pi x_1)\sin^2(m\pi x_2)\, dx_1\, dx_2 \\
&= \int_0^1 \frac{1-\cos(2n\pi x_1)}{2}\, dx_1 \cdot \int_0^1 \frac{1-\cos(2m\pi x_2)}{2}\, dx_2 \\
&= \left[x_1/2 - \frac{\sin(2n\pi x_1)}{4n\pi}\right]_0^1 \cdot \left[x_2/2 - \frac{\sin(2m\pi x_2)}{4m\pi}\right]_0^1
\end{aligned}$$

$$= \left(1/2 - \frac{\sin(2n\pi)}{4n\pi}\right) \cdot \left(1/2 - \frac{\sin(2m\pi)}{4m\pi}\right) = 1/4,$$

and thus $\|f\|_{L^2(\Omega)} = 1/2$ if $n \neq 0$ and $m \neq 0$.

Problem 3. Let $\mathcal{P}(K) = \{v(x) = c_0 + c_1x_1 + c_2x_2,\ c_i \in \mathbf{R},\ i = 1,2,3;\ x = (x_1, x_2) \in K\}$ be the space of linear polynomials defined on a triangle K with corners a^1, a^2, and a^3. Derive explicit expressions (in terms of the corner coordinates $a^1 = (a_1^1,\ a_2^1)$, $a^2 = (a_1^2,\ a_2^2)$, and $a^3 = (a_1^3,\ a_2^3)$) for the basis functions λ_1, λ_2, $\lambda_3 \in \mathcal{P}(K)$ defined by

$$\lambda_i(a^j) = \begin{cases} 1 & i = j, \\ 0 & i \neq j, \end{cases} \tag{14}$$

with $i, j = 1, 2, 3$. Hint: set up the linear system of equations which relates c_0, c_1, and c_2 to the values at the corners $v(a^1)$, $v(a^2)$, and $v(a^3)$ of a function $v \in \mathcal{P}(K)$. Solve for the coefficients corresponding to corner values of the basis functions.

Solution: Look at the basis function λ_1 first. Since λ_1 is *linear* on K we make the Ansatz $\lambda_1(x_1,\ x_2) = c_0 + c_1x_1 + c_2x_2$. According to the definition λ_1 has the value one in a^1 and zero in a^2 and a^3. (See Figure 10.) Hence, we have in these corners respectively:

$$\begin{cases} 1 = c_0 + c_1a_1^1 + c_2a_2^1 \\ 0 = c_0 + c_1a_1^2 + c_2a_2^2 \\ 0 = c_0 + c_1a_1^3 + c_2a_2^3 \end{cases}$$

Or in matrix form:

$$\underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{b} = \underbrace{\begin{pmatrix} 1 & a_1^1 & a_2^1 \\ 1 & a_1^2 & a_2^2 \\ 1 & a_1^3 & a_2^3 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix}}_{c}$$

We have three equations and three unknowns (c_0, c_1 and c_2). We can solve the linear system of equations above by Gaussian elimination. The result is

$$\begin{aligned} c_0 &= \frac{a_1^2a_2^3 - a_1^3a_2^2}{\det A} \\ c_1 &= \frac{a_2^2 - a_2^3}{\det A} \\ c_2 &= \frac{a_1^3 - a_1^2}{\det A} \end{aligned}$$

where $\det A = a_1^3a_2^1 + a_1^2a_2^3 - a_1^2a_2^1 - a_1^3a_2^2 - a_1^1a_2^3 + a_1^1a_2^2$.

For the basis function λ_2 we get the same matrix A as above, but here $b = (0,\ 1,\ 0)^T$ (since λ_2 is one in the node a^2 and zero in the other two nodes). Solving the system of equations gives

$$
\begin{aligned}
c_0 &= \frac{a_1^3 a_2^1 - a_1^1 a_2^3}{\det A} \\
c_1 &= \frac{a_2^3 - a_2^1}{\det A} \\
c_2 &= \frac{a_1^1 - a_1^3}{\det A}
\end{aligned}
$$

And similarly for λ_3 with $b = (0,\, 0,\, 1)^T$ gives the coefficients

$$
\begin{aligned}
c_0 &= \frac{a_1^1 a_2^2 - a_1^2 a_2^1}{\det A} \\
c_1 &= \frac{a_2^1 - a_2^2}{\det A} \\
c_2 &= \frac{a_1^2 - a_1^1}{\det A}
\end{aligned}
$$

Remark. Note that $\det A$ equals $2\,\mu(K)$ where $\mu(K)$ is the area of K. See *Problem 4 (Week 6)*. Note further that it might not be necessary to actually compute λ_2 and λ_3. Given the expression for λ_1 it is possible to make a permutation of the node indices.

□

Problem 4. Derive an expression for the area of the triangle K in *Problem 3 (Week 6)* in terms of the corner coordinates $a^1 = (a_1^1,\, a_2^1)$, $a^2 = (a_1^2,\, a_2^2)$ and $a^3 = (a_1^3,\, a_2^3)$.
Solution:

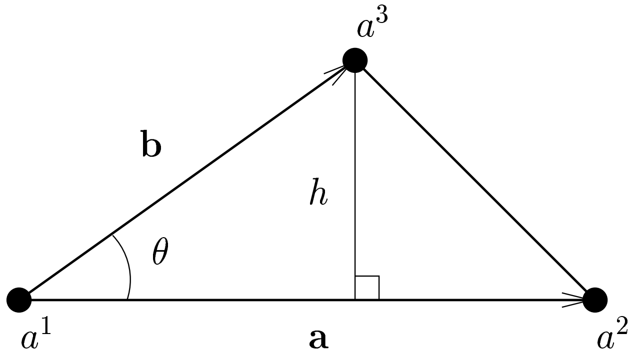

Figure 10: Problem 3 and Problem 4 (Week 6).

From Figure 10 we calculate the area $\mu(K)$ as follows.

$$
\mu(K) = \tfrac{1}{2}\,|\mathbf{a}|h = \tfrac{1}{2}\,|\mathbf{a}||\mathbf{b}|\sin\theta = \tfrac{1}{2}|\mathbf{a}\times\mathbf{b}| \tag{15}
$$

Now, clearly the vectors $\mathbf{a}$ and $\mathbf{b}$ are given by

$$\mathbf{a} = a^2 - a^1 = (a_1^2 - a_1^1,\, a_2^2 - a_2^1), \tag{16}$$
$$\mathbf{b} = a^3 - a^1 = (a_1^3 - a_1^1,\, a_2^3 - a_2^1). \tag{17}$$

Explicitly the area is thus given by

$$\mu(K) = \tfrac{1}{2}|\mathbf{a} \times \mathbf{b}| = | \begin{vmatrix} a_1^2 - a_1^1 & a_2^2 - a_2^1 \\ a_1^3 - a_1^1 & a_2^3 - a_2^1 \end{vmatrix} | \tag{18}$$
$$= \tfrac{1}{2}|(a_1^2 - a_1^1)(a_2^3 - a_2^1) - (a_2^2 - a_2^1)(a_1^3 - a_1^1)|. \tag{19}$$

Note that the cross-product between vectors in two dimensions is a number.

Remark. With $\mathbf{a}$ and $\mathbf{b}$ oriented as in Figure 10 the cross-product $\mathbf{a} \times \mathbf{b}$ is positive and thus $\mu(K) = \frac{1}{2}(\mathbf{a} \times \mathbf{b})$.

□

Problem 5. Consider the triangulation of $\Omega = [0, 2] \times [0, 1]$ into 3 triangles drawn in Figure 11.

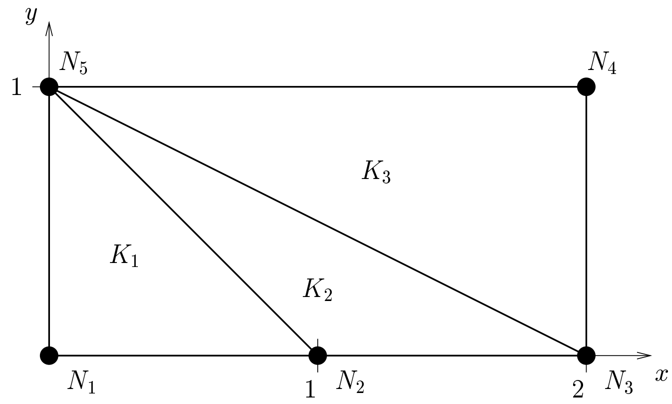

Figure 11: Problem 5 (Week 6). The triangulation of Ω.

(a) Compute the mass matrix M with elements $m_{ij} = \iint_\Omega \varphi_j(x, y)\, \varphi_i(x, y)\, dx\, dy$, $i, j = 1, \ldots, 5$.

Hint: The easiest way is to use the quadrature formula based on the value of the integrand, $\varphi_j(x, y)\, \varphi_i(x, y)$, at the mid-points on the triangle sides, since this formula is exact for polynomials of degree 2. It is also possible to write down explicit analytical expressions for the "tent-functions" on each triangle (cf. *Problem 3 (Week 6)*) and integrate the products analytically. This, however, is a much harder way. Observe that, using quadrature, we don't need to know the analytical expressions, only *the values at some given points* which are much easier to compute.

(b) Compute the "lumped" mass matrix $\hat{M}$, which is the diagonal matrix with the diagonal element in each row being the sum of the elements in the corresponding row of M.
(c*) Prove that, using nodal quadrature, the approximate mass matrix you get is actually the "lumped" mass matrix.

Hint: $\sum_{j=1}^{5} \varphi_j(x,\, y) \equiv 1$

Solution:
(a) We start to compute the area $\mu(K_i)$ of the triangles, $i = 1, 2, 3$:

$$\mu(K_1) = \frac{1 \cdot 1}{2} = \frac{1}{2},$$

$$\mu(K_2) = \frac{1 \cdot 1}{2} = \frac{1}{2},$$

$$\mu(K_3) = \frac{2 \cdot 1}{2} = 1.$$

Then, we compute a few elements of M: m_{11}, m_{12}, m_{13}, and m_{22}. Note that the integrands $\varphi_1\,\varphi_1$ and $\varphi_2\,\varphi_1$ are non-zero only over K_1, and $\varphi_2\,\varphi_2$ is non-zero over K_1 and K_2. On the other hand $\varphi_3\,\varphi_1$ is nowhere non-zero and therefore $m_{13} = 0$.

$$\begin{aligned} m_{11} = \iint_\Omega \varphi_1\,\varphi_1\,dxdy &= \frac{(\varphi_1(\frac{1}{2},0))^2 + (\varphi_1(0,\frac{1}{2}))^2 + (\varphi_1(\frac{1}{2},\frac{1}{2}))^2}{3}\,\mu(K_1) \\ &= \frac{\frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} + 0\cdot 0}{3}\,\mu(K_1) = \frac{1}{6}\,\mu(K_1) = \frac{1}{12}, \end{aligned}$$

$$m_{12} = (M \text{ symmetric!}) = m_{21} = \frac{\frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot 0 + 0\cdot\frac{1}{2}}{3}\,\mu(K_1) = \frac{1}{12}\,\mu(K_1) = \frac{1}{24},$$

$$m_{22} = \frac{\frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} + 0}{3}\,\mu(K_1) + \frac{\frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} + 0}{3}\,\mu(K_2) = \frac{1}{6}\,(\mu(K_1) + \mu(K_2)) = \frac{1}{6}.$$

Continuing analogously gives:

$$M = \begin{bmatrix} \frac{1}{12} & \frac{1}{24} & 0 & 0 & \frac{1}{24} \\ \frac{1}{24} & \frac{1}{6} & \frac{1}{24} & 0 & \frac{1}{12} \\ 0 & \frac{1}{24} & \frac{1}{4} & \frac{1}{12} & \frac{1}{8} \\ 0 & 0 & \frac{1}{12} & \frac{1}{6} & \frac{1}{12} \\ \frac{1}{24} & \frac{1}{12} & \frac{1}{8} & \frac{1}{12} & \frac{1}{3} \end{bmatrix}$$

(b) From $\hat{m}_{ii} = \sum_{j=1}^{5} m_{ij}, \quad i = 1, \ldots, 5$, we compute:

$$\hat{m}_{11} = \frac{1}{12} + \frac{1}{24} + 0 + 0 + \frac{1}{24} = \frac{1}{6}.$$

Analogously:

$$\hat{m}_{22} = \frac{1}{3}; \quad \hat{m}_{33} = \frac{1}{2}; \quad \hat{m}_{44} = \frac{1}{3}; \quad \hat{m}_{55} = \frac{2}{3}.$$

Thus:

$$\hat{M} = \begin{bmatrix} \frac{1}{6} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & \frac{2}{3} \end{bmatrix}$$

(c*) *Hint:* Adding the elements in row number i gives:

$$\hat{m}_{ii} = \iint_\Omega \left(\sum_{j=1}^{5} \varphi_j(x,\, y) \right) \varphi_i(x,\, y)\, dx\, dy = \iint_\Omega \varphi_i(x,\, y)\, dx\, dy.$$

Now use the formula for the volume of a pyramid, and compare the result to what you get when using nodal quadrature. □

Week 7:

Problem 1. Compute ∇u, $n \cdot \nabla u$, and $\triangle u$ for
(a) $u(x,\, y) = xy; \quad n = (1,\, 0)$,
(b) $u(x,\, y) = \sin(x)\cos(y); \quad n = (1,\, 1)$,
(c) $u(x,\, y) = \log(r)$ where $r = \sqrt{x^2 + y^2} \quad (r \neq 0); \quad n = (x,\, y)$.
Solution:
(a) $\nabla u = (\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}) = (y, x)$, so $n \cdot \nabla u = y$, and $\triangle u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial y}{\partial x} + \frac{\partial x}{\partial y} = 0$.
(b) $\nabla u = (\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}) = (\cos(x)\cos(y), -\sin(x)\sin(y))$, so
$n \cdot \nabla u = \cos(x)\cos(y) - \sin(x)\sin(y) = \cos(x+y)$, and
$\triangle u = \frac{\partial(\cos(x)\cos(y))}{\partial x} - \frac{\partial(\sin(x)\sin(y))}{\partial y} = -\sin(x)\cos(y) - \sin(x)\cos(y) = -2\sin(x)\cos(y)$.
(c) $\nabla u = (\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}) = (\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}) = \frac{1}{r^2}(x, y)$, so $n \cdot \nabla u = \frac{1}{r^2}(x^2 + y^2) = 1$, and
$\triangle u = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) = 0$. $\square$

Problem 2. Consider the triangulation of $\Omega = [0,\, 2] \times [0,\, 1]$ into 3 triangles drawn in Figure 12. (It is the same triangulation as in *Problem 5 (Week 6)*.)

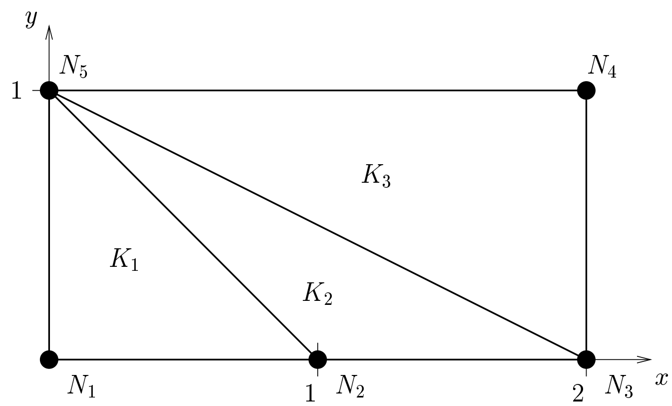

Figure 12: Problem 1 and Problem 4 (Week 7). The triangulation of Ω.

Compute by hand the stiffness matrix A with elements $a_{ij} = \iint_\Omega \nabla\varphi_j \cdot \nabla\varphi_i \, dx\, dy$, $i, j = 1, \ldots, 5$.

Hint: Since $\varphi_i(x,\, y)$ is linear on each triangle, the gradient $\nabla\varphi_i$ will be a *constant* vector on each triangle. As an example, consider triangle K_1. On this triangle, it is easy to show that $\varphi_1(x,\, y) = 1 - (x + y)$, $\varphi_2(x,\, y) = x$, and $\varphi_5(x,\, y) = y$ (cf. how you did in *Problem 2(a) (Week 5)*). Therefore, on K_1: $\nabla\varphi_1 = (-1, -1)$, $\nabla\varphi_2 = (1,\, 0)$, and $\nabla\varphi_5 = (0,\, 1)$. Thus, $a_{11} = \iint_\Omega \nabla\varphi_1 \cdot \nabla\varphi_1 \, dx\, dy = \iint_{K_1} \nabla\varphi_1 \cdot \nabla\varphi_1 \, dx\, dy = \iint_{K_1} 2\, dx\, dy = 1$. Observe that some matrix elements will get contributions from more than one triangle.
Solution: The matrix A with elements $a_{ij} = \iint \nabla\varphi_j \cdot \nabla\varphi_i \, dxdy$ is clearly symmetric. One easily sees which elements in A that are zero. For example, since φ_1 only is non-zero on triangle K_1 and φ_4 only is non-zero on triangle K_3, we know that $a_{14} = a_{41} = 0$. Similarly we see that $a_{13} = a_{31} = a_{24} = a_{42} = 0$.

Since $\varphi_i(x,y)$ is linear on each triangle, the gradient $\nabla\varphi_i$ will be a constant vector on each triangle.

We now calculate a_{55}. The function φ_5 is non-zero on all triangles. By solving a linear system of equations on each triangle (cf. *Problem 2(a) (Week 5)*), we get that $\varphi_5(x,y) = y$ on triangles K_1 and K_2, and $\varphi_5(x,y) = 1 - x/2$ on triangle K_3. So $\nabla\varphi_5 = (0,1)$ on triangles K_1 and K_2, and $\nabla\varphi_5 = (-1/2, 0)$ on triangle K_3. Thus,

$$\begin{aligned} a_{55} &= \iint_\Omega \nabla\varphi_5 \cdot \nabla\varphi_5 \, dxdy = \iint_{K_1 \cup K_2} 1 \, dxdy + \iint_{K_3} 1/4 \, dxdy \\ &= \mu(K_1 \cup K_2) + \frac{1}{4}\mu(K_3) = 1 + 1/4 = 5/4, \end{aligned}$$

where $\mu(K_1 \cup K_2) = 1$ and $\mu(K_3) = 1$ denote the areas of $K_1 \cup K_2$ and K_3 respectively.

We now calculate $a_{12} = a_{21}$. Since $\varphi_1(x,y)$ only is non-zero on triangle K_1 it is enough to integrate over triangle K_1, where $\nabla\varphi_1 = (-1,-1)$ and $\nabla\varphi_2 = (1,0)$ (see the given Hint in the exercise):

$$\begin{aligned} a_{12} = a_{21} &= \iint_\Omega \nabla\varphi_2 \cdot \nabla\varphi_1 \, dxdy = \iint_{K_1} \nabla\varphi_2 \cdot \nabla\varphi_1 \, dxdy \\ &= \iint_{K_1} -1 \, dxdy = -\mu(K_1) = -1/2. \end{aligned}$$

Similarly we now calculate a_{22}. Since $\varphi_2(x,y)$ only is non-zero on triangles K_1 and K_2 it is enough to integrate over these triangles. On K_1, $\varphi_2(x,y) = x$ so there $\nabla\varphi_2 = (1,0)$, and on K_2, $\varphi_2(x,y) = 2 - (x+2y)$ so there $\nabla\varphi_2 = (-1,-2)$. This gives that

$$\begin{aligned} a_{22} &= \iint_{K_1} \nabla\varphi_2 \cdot \nabla\varphi_2 \, dxdy + \iint_{K_2} \nabla\varphi_2 \cdot \nabla\varphi_2 \, dxdy \\ &= \iint_{K_1} 1 \, dxdy + \iint_{K_2} 5 \, dxdy = 1/2 + 5/2 = 3. \end{aligned}$$

In the same way as above one gets that $a_{11} = 1$, $a_{33} = 2$, $a_{44} = \frac{5}{4}$, $a_{43} = a_{34} = -1$, $a_{15} = a_{51} = -\frac{1}{2}$, $a_{25} = a_{52} = -1$, $a_{23} = a_{32} = -\frac{3}{2}$, $a_{35} = a_{53} = \frac{1}{2}$ and $a_{45} = a_{54} = -\frac{1}{4}$. Thus:

$$A = \begin{bmatrix} 1 & -1/2 & 0 & 0 & -1/2 \\ -1/2 & 3 & -3/2 & 0 & -1 \\ 0 & -3/2 & 2 & -1 & 1/2 \\ 0 & 0 & -1 & 5/4 & -1/4 \\ -1/2 & -1 & 1/2 & -1/4 & 5/4 \end{bmatrix}.$$

□

Problem 3. Let $\mathcal{P}(K) = \{v(x) = c_0 + c_1x_1 + c_2x_2,\ c_i \in \mathbf{R},\ i = 1,2,3;\ x = (x_1, x_2) \in K\}$ be the space of linear polynomials defined on a triangle K with corners a^1, a^2, and a^3. Derive explicit expressions (in terms of the corner coordinates $a^1 = (a_1^1,\, a_2^1)$, $a^2 = (a_1^2,\, a_2^2)$, and $a^3 = (a_1^3,\, a_2^3))$ for the gradients $\nabla\lambda_1$, $\nabla\lambda_2$, $\nabla\lambda_3$ of the basis functions $\lambda_1, \lambda_2, \lambda_3 \in \mathcal{P}(K)$ defined by

$$\lambda_i(a^j) = \begin{cases} 1 & i = j, \\ 0 & i \neq j, \end{cases}$$

with $i, j = 1, 2, 3$. Compare with the corresponding expressions in your Matlab-function `MyFirst2DPoissonAssembler`.

Hint: Use the result from *Problem 3 (Week 6).*

Solution: Since $v(x) = c_0 + c_1x_1 + c_2x_2$, we have $\nabla v(x) = (c_1, c_2)$. All we have to do to determine the gradients is then to identify the coefficients c_1 and c_2. From *Problem 3 (Week 6)*, we get

$$\begin{aligned}
\nabla\lambda_1 &= \frac{1}{\det A}(a_2^2 - a_2^3,\, a_1^3 - a_1^2), \\
\nabla\lambda_2 &= \frac{1}{\det A}(a_2^3 - a_2^1,\, a_1^1 - a_1^3), \\
\nabla\lambda_2 &= \frac{1}{\det A}(a_2^1 - a_2^2,\, a_1^2 - a_1^1),
\end{aligned}$$

where $\det A = a_1^3a_2^1 + a_1^2a_2^3 - a_1^2a_2^1 - a_1^3a_2^2 - a_1^1a_2^3 + a_1^1a_2^2 = 2\,\mu(K)$, where $\mu(K)$ is the area of K. (Cf. *Problem 4 (Week 6).*) Note that the gradients are constant, which is a property of a plane. □

Problem 4. Consider once more the triangulation of $\Omega = [0,\, 2] \times [0,\, 1]$ into 3 triangles drawn in Figure 12. Let $\Gamma = \partial\Omega$ denote the boundary of Ω. Assuming that $\gamma(x,\, y) = 1$, $g_D(x,\, y) = 1 + x + y$, and $g_N(x,\, y) = 0$, compute by hand:
(a) The "boundary matrix" R with elements $r_{ij} = \int_\Gamma \gamma\,\varphi_j\,\varphi_i\,ds$, $i, j = 1, \ldots, 5$.
(b) The "boundary vector" rv with elements $\mathrm{rv}_i = \int_\Gamma (\gamma g_D - g_N)\,\varphi_i\,ds$, $i = 1, \ldots, 5$.

Hint: You can either compute the curve integrals analytically or use *Simpson's* formula which is exact in this case.

Solution: Start by dividing and numbering the boundary Γ into five segments Γ_i, $i = 1, \ldots, 5$ according to Figure 13.
(a) The first row of the "boundary matrix" is now computed using Simpson's rule. When doing this we have to keep track of where the basis functions are non-zero. For instance, φ_1 is identically equal to zero on the boundary except on the segments Γ_1 and Γ_5. Further, the value of φ_1 at the midpoints of Γ_1 and Γ_5 is $\frac{1}{2}$. Since $\gamma = 1$ this gives:

$$\begin{aligned}
r_{11} &= \int_\Gamma \varphi_1^2\,ds = \int_{\Gamma_5} \varphi_1^2\,ds + \int_{\Gamma_1} \varphi_1^2\,ds = \int_0^1 \varphi_1(0,\, y)^2\,dy + \int_0^1 \varphi_1(x,\, 0)^2\,dx \\
&= \{\text{Simpson's rule}\} = \frac{1\cdot 1 + 4\cdot\frac{1}{2}\cdot\frac{1}{2} + 0\cdot 0}{6}\cdot 1 + \frac{1\cdot 1 + 4\cdot\frac{1}{2}\cdot\frac{1}{2} + 0\cdot 0}{6}\cdot 1 = \frac{2}{3}.
\end{aligned}$$

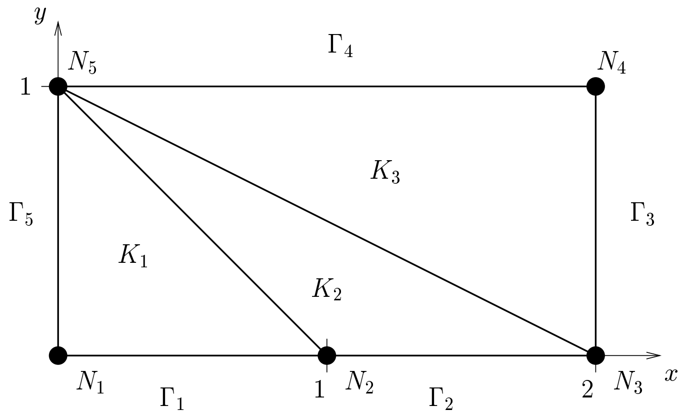

Figure 13: Problem 4 (Week 7). The five segments of Γ.

Since φ_1 and φ_2 are non-zero simultaneously only on Γ_1 we get:

$$r_{12} = \int_{\Gamma_1} \varphi_2 \varphi_1 \, ds = \int_0^1 \varphi_2(x,\, 0)\varphi_1(x,\, 0)\, dx = \frac{0 \cdot 1 + 4 \cdot \frac{1}{2} \cdot \frac{1}{2} + 1 \cdot 0}{6} \cdot 1 = \frac{1}{6},$$

and analogously:

$$r_{15} = \int_{\Gamma_5} \varphi_5 \varphi_1 \, ds = \int_0^1 \varphi_5(0,\, y)\varphi_1(0,\, y)\, dy = \frac{0 \cdot 1 + 4 \cdot \frac{1}{2} \cdot \frac{1}{2} + 1 \cdot 0}{6} \cdot 1 = \frac{1}{6}.$$

The matrix element $r_{13} = 0$ since φ_1 and φ_3 don't overlap on any boundary segment. The same reasoning leads to $r_{14} = 0$. Similar computations give the rest of the matrix elements (also note that R is symmetric), we just have to remember how long the boundary segments are. (Be careful with the integrals over Γ_4; don't forget that this segment has length 2!) The final result is:

$$R = \begin{bmatrix} \frac{2}{3} & \frac{1}{6} & 0 & 0 & \frac{1}{6} \\ & \frac{2}{3} & \frac{1}{6} & 0 & 0 \\ & & \frac{2}{3} & \frac{1}{6} & 0 \\ & \text{symm.} & & 1 & \frac{1}{3} \\ & & & & 1 \end{bmatrix}.$$

(b) We start by computing the first component rv_1 of the "boundary vector" rv. Since $\gamma = 1$, $g_D = 1 + x + y$ and $g_N = 0$ the integrand becomes $(1 + x + y)\,\varphi_1(x,\, y)$. Note that since φ_1 is non-zero only on Γ_1 and Γ_5 we need only integrate over these two boundary segments. Further note that $y = 0$ on Γ_1 and that $x = 0$ on Γ_5:

$$\mathrm{rv}_1 = \int_\Gamma (1 + x + y)\,\varphi_1(x,\, y)\, ds = \int_{\Gamma_1} (1 + x + 0)\,\varphi_1(x,\, 0)\, ds + \int_{\Gamma_5} (1 + 0 + y)\,\varphi_1(0,\, y)\, ds$$

$$=\{ds = dx \text{ on } \Gamma_1;\, ds = dy \text{ on } \Gamma_5\} = \int_0^1 (1+x)\,\varphi_1(x,\,0)\,dx + \int_0^1 (1+y)\,\varphi_1(0,\,y)\,dy$$
$$=\{\text{Simpson's rule}\} = \frac{1\cdot 1 + 4\cdot\frac{3}{2}\cdot\frac{1}{2} + 2\cdot 0}{6}\cdot 1 + \frac{1\cdot 1 + 4\cdot\frac{3}{2}\cdot\frac{1}{2} + 2\cdot 0}{6}\cdot 1$$
$$=\frac{4}{6} + \frac{4}{6} = \frac{4}{3}.$$

Continuing in the same way gives the rest of the elements:

$$\mathrm{rv}_2 = \int_\Gamma (1+x+y)\,\varphi_2(x,\,y)\,ds = \int_{\Gamma_1} (1+x+0)\,\varphi_2(x,\,0)\,ds + \int_{\Gamma_2} (1+x+0)\,\varphi_2(x,\,0)\,ds$$
$$=\{ds = dx \text{ on } \Gamma_1 \cup \Gamma_2\} = \int_0^1 (1+x)\,\varphi_2(x,\,0)\,dx + \int_1^2 (1+x)\,\varphi_2(x,\,0)\,dx$$
$$=\{\text{Simpson's rule}\} = \frac{1\cdot 0 + 4\cdot\frac{3}{2}\cdot\frac{1}{2} + 2\cdot 1}{6}\cdot 1 + \frac{2\cdot 1 + 4\cdot\frac{5}{2}\cdot\frac{1}{2} + 3\cdot 0}{6}\cdot 1$$
$$=\frac{5}{6} + \frac{7}{6} = 2,$$

$$\mathrm{rv}_3 = \int_\Gamma (1+x+y)\,\varphi_3(x,\,y)\,ds = \int_{\Gamma_2} (1+x+0)\,\varphi_3(x,\,0)\,ds + \int_{\Gamma_3} (1+2+y)\,\varphi_3(2,\,y)\,ds$$
$$=\{ds = dx \text{ on } \Gamma_2;\, ds = dy \text{ on } \Gamma_3\} = \int_1^2 (1+x)\,\varphi_3(x,\,0)\,dx + \int_0^1 (3+y)\,\varphi_3(2,\,y)\,dy$$
$$=\{\text{Simpson's rule}\} = \frac{2\cdot 0 + 4\cdot\frac{5}{2}\cdot\frac{1}{2} + 3\cdot 1}{6}\cdot 1 + \frac{3\cdot 1 + 4\cdot\frac{7}{2}\cdot\frac{1}{2} + 4\cdot 0}{6}\cdot 1$$
$$=\frac{8}{6} + \frac{10}{6} = 3,$$

$$\mathrm{rv}_4 = \int_\Gamma (1+x+y)\,\varphi_4(x,\,y)\,ds = \int_{\Gamma_3} (1+2+y)\,\varphi_4(2,\,y)\,ds + \int_{\Gamma_4} (1+x+1)\,\varphi_4(x,\,1)\,ds$$
$$=\{ds = dy \text{ on } \Gamma_3;\, ds = dx \text{ on } \Gamma_4\} = \int_0^1 (3+y)\,\varphi_4(2,\,y)\,dy + \int_0^2 (2+x)\,\varphi_4(x,\,1)\,dx$$
$$=\{\text{Simpson's rule}\} = \frac{3\cdot 0 + 4\cdot\frac{7}{2}\cdot\frac{1}{2} + 4\cdot 1}{6}\cdot 1 + \frac{2\cdot 0 + 4\cdot 3\cdot\frac{1}{2} + 4\cdot 1}{6}\cdot 2$$
$$=\frac{11}{6} + \frac{20}{6} = 5\frac{1}{6},$$

$$\mathrm{rv}_5 = \int_\Gamma (1+x+y)\,\varphi_5(x,\,y)\,ds = \int_{\Gamma_4} (1+x+1)\,\varphi_5(x,\,1)\,ds + \int_{\Gamma_5} (1+0+y)\,\varphi_5(0,\,y)\,ds$$
$$=\{ds = dx \text{ on } \Gamma_4;\, ds = dy \text{ on } \Gamma_5\} = \int_0^2 (2+x)\,\varphi_5(x,\,1)\,dx + \int_0^1 (1+y)\,\varphi_5(0,\,y)\,dy$$

$$
\begin{aligned}
&=\{\text{Simpson's rule}\} = \frac{2\cdot 1 + 4\cdot 3\cdot \frac{1}{2} + 4\cdot 0}{6}\cdot 2 + \frac{1\cdot 0 + 4\cdot \frac{3}{2}\cdot \frac{1}{2} + 2\cdot 1}{6}\cdot 1 \\
&=\frac{16}{6} + \frac{5}{6} = 3\frac{1}{2}.
\end{aligned}
$$

□

Problem 5. Show that the equation:

$$\iint_\Omega \nabla U \cdot \nabla v \, dx\, dy = \iint_\Omega f v \, dx\, dy \quad \text{for all } v \in V_{h0}, \tag{20}$$

is equivalent to

$$\iint_\Omega \nabla U \cdot \nabla \varphi_i \, dx\, dy = \iint_\Omega f \varphi_i \, dx\, dy \quad \text{for } i = 1, \ldots, N, \tag{21}$$

where N is the number of internal nodes ("*nintnodes*") and $\{\varphi_i\}_{i=1}^N$ is the basis of "tent-functions" in V_{h0}.

Solution:

⇒: We assume that (20) is true:

$$\iint_\Omega \nabla U \cdot \nabla v \, dxdy = \iint_\Omega f v \, dxdy \quad \forall\, v \in V_{h0},$$

and want to show that this implies that (21) is true:

$$\iint_\Omega \nabla U \cdot \nabla \varphi_i \, dxdy = \iint_\Omega f \varphi_i \, dxdy \quad i = 1, \ldots, N.$$

But since (20) holds for all $v \in V_{h0}$ and $\varphi_i \in V_{h0}$, $i = 1, \ldots, N$, it´s for sure that (20) implies (21).

⇐: We now assume that (21) is true:

$$\iint_\Omega \nabla U \cdot \nabla \varphi_i \, dxdy = \iint_\Omega f \varphi_i \, dxdy \quad i = 1, \ldots, N,$$

and want to show that this implies that (20) is true:

$$\iint_\Omega \nabla U \cdot \nabla v \, dxdy = \iint_\Omega f v \, dxdy \quad \forall\, v \in V_{h0}.$$

Since $\{\varphi_i\}_{i=1}^N$ is a basis for V_{h0}, every $v \in V_{h0}$ can be written:

$$v(x,\, y) = \sum_{i=1}^N c_i\, \varphi_i(x,\, y)$$

for some constants c_i, $i = 1, \dots, N$. Multiply (21) with arbitrary constants c_i for all $i = 1, \dots, N$. If we add it all together, we get:

$$\iint_\Omega \nabla U \cdot \nabla(c_1\varphi_1)\, dxdy + \iint_\Omega \nabla U \cdot \nabla(c_2\varphi_2)\, dxdy + \cdots + \iint_\Omega \nabla U \cdot \nabla(c_N\varphi_N)\, dxdy$$

$$= \iint_\Omega f c_1\varphi_1\, dxdy + \iint_\Omega f c_2\varphi_2\, dxdy + \cdots + \iint_\Omega f c_N\varphi_N\, dxdy,$$

and hence:

$$\iint_\Omega \nabla U \cdot \nabla(\sum_{i=1}^{N} c_i\varphi_i)\, dxdy = \iint_\Omega f(\sum_{i=1}^{N} c_i\varphi_i)\, dxdy, \tag{22}$$

for arbitrary constants c_i, $i = 1, \dots, N$. Since (22) holds for *every* set of constants $\{c_i\}_{i=1}^N$, we conclude that (20) holds for all $v \in V_{h0}$. □

Problem 6*. Show that the problem: find $U \in V_{h0}$ such that

$$\iint_\Omega \nabla U \cdot \nabla w dx\, dy = \iint_\Omega f w\, dx\, dy \quad \text{for all } w \in V_{h0}, \tag{23}$$

is equivalent to the minimization problem: find $U \in V_{h0}$ such that

$$\frac{1}{2}\iint_\Omega \nabla U \cdot \nabla U\, dx\, dy - \iint_\Omega fU\, dx\, dy = \min_{v \in V_{h0}} \frac{1}{2}\iint_\Omega \nabla v \cdot \nabla v\, dx\, dy - \iint_\Omega f v\, dx\, dy. \tag{24}$$

Solution:

□

Problem 7*.
(a) Consider the quadratic equation

$$at^2 + bt + c = 0, \tag{25}$$

Investigate under what condition on the coefficients a, b, c equation (25) does *not* have two distinct real roots.
(b) Prove the Cauchy-Schwarz inequality:

$$|\iint_\Omega vw\, dx\, dy| \leq \|v\|_{L^2(\Omega)}\|w\|_{L^2(\Omega)} \tag{26}$$

Hint: start from the fact that $\|v+tw\|^2_{L^2(\Omega)} \geq 0$. Expanding $\|v+tw\|^2_{L^2(\Omega)}$ gives a quadratic polynomial which can not have two distinct real roots (why?). Use (a) to prove the Cauchy-Schwarz inequality.
Solution:

□

Problem 8. Calculate $\|\nabla f\|_{L^2(\Omega)}$ where $\Omega = [0,1] \times [0,1]$ and
(a) $f = x_1 x_2^2$.
(b) $f = \sin(nx_1)\sin(mx_2)$ with n and m arbitrary integers. What happens when n, m tends to infinity?
Solution:
(a) Recall that the gradient, ∇f, of a scalar function $f(x_1, x_2)$ is a vector with the partial derivatives of f as components:

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}\right) = (x_2^2,\ 2x_1 x_2).$$

Now, using the definition of the L^2-norm we get:

$$\begin{aligned}
\|\nabla f\|_{L^2(\Omega)}^2 &= \iint_\Omega |\nabla f|^2 \, dx_1 dx_2 = \iint_\Omega \nabla f \cdot \nabla f \, dx_1 dx_2 = \iint_\Omega x_2^4 + 4x_1^2 x_2^2 \, dx_1 dx_2 \\
&= \int_0^1 \int_0^1 (x_2^4 + 4x_1^2 x_2^2) \, dx_2 dx_1 = \int_0^1 \left[\tfrac{1}{5}x_2^5 + \tfrac{4}{3}x_1^2 x_2^3\right]_0^1 dx_1 \\
&= \int_0^1 \left(\tfrac{1}{5} + \tfrac{4}{3}x_1^2\right) dx_1 = \left[\tfrac{1}{5}x_1 + \tfrac{4}{9}x_1^3\right]_0^1 = \frac{1}{5} + \frac{4}{9} = \frac{29}{45},
\end{aligned}$$

i.e. we have the answer $\|\nabla f\|_{L^2(\Omega)} = \sqrt{\frac{29}{45}}$.
(b) First note that if $n = 0$ and/or $m = 0$ we have $f \equiv 0$ and therefore $\nabla f = (0, 0)$ and $\|\nabla f\|_{L^2(\Omega)} = 0$. If $n \neq 0$ and $m \neq 0$ we first, as in (a), compute the gradient vector ∇f of the function $f(x_1, x_2)$:

$$\nabla f = (n\cos(nx_1)\sin(mx_2),\ m\sin(nx_1)\cos(mx_2)).$$

As above, we then compute:

$$\|\nabla f\|_{L^2(\Omega)}^2 = \iint_\Omega \left(n^2\cos^2(nx_1)\sin^2(mx_2) + m^2\sin^2(nx_1)\cos^2(mx_2)\right) dx_1 dx_2.$$

This looks a bit nasty but using the well known trigonometric formulas

$$\sin^2\alpha = \tfrac{1}{2}(1 - \cos 2\alpha), \quad \cos^2\alpha = \tfrac{1}{2}(1 + \cos 2\alpha)$$

we can rewrite the integral, which then equals:

$$\iint_\Omega \left(\frac{n^2}{4}(1 + \cos 2nx_1)(1 - \cos 2mx_2) + \frac{m^2}{4}(1 - \cos 2nx_1)(1 + \cos 2mx_2)\right) dx_1 dx_2.$$

Since the factors in the integrand are independent of each other we can break the integral into four separate and simple parts. We thus have:

$$\begin{aligned}
\|\nabla f\|^2_{L^2(\Omega)} &= \frac{n^2}{4}\int_0^1 (1+\cos 2nx_1)\,dx_1 \int_0^1 (1-\cos 2mx_2)\,dx_2 \\
&+ \frac{m^2}{4}\int_0^1 (1-\cos 2nx_1)\,dx_1 \int_0^1 (1+\cos 2mx_2)\,dx_2 \\
&= \frac{n^2}{4}\left[x_1 + \frac{\sin 2nx_1}{2n}\right]_0^1 \left[x_2 - \frac{\sin 2mx_1}{2m}\right]_0^1 + \frac{m^2}{4}\left[x_1 - \frac{\sin 2nx_1}{2n}\right]_0^1 \left[x_2 + \frac{\sin 2mx_2}{2m}\right]_0^1 \\
&= \frac{n^2}{4}\left(1+\frac{\sin 2n}{2n}\right)\left(1-\frac{\sin 2m}{2m}\right) + \frac{m^2}{4}\left(1-\frac{\sin 2n}{2n}\right)\left(1+\frac{\sin 2m}{2m}\right).
\end{aligned}$$

If we let n, m tend to infinity the terms involving *sine* tend to zero because of the big terms in the denominators and we are left with

$$\|\nabla f\|^2_{L^2(\Omega)} \sim \frac{n^2}{4} + \frac{m^2}{4} \to \infty \quad n, m \to \infty.$$

This can be understood if we consider the effect of n in the expression $\sin nx_1$. The integer n determines how fast the function will oscillate, i.e., the frequency. As n tends to infinity the function will oscillate increasingly faster, causing its derivative to become large. And since the norm is a measure of the gradient's size, it will become infinite in the limit. □

Problem 9. Let $u = x_1x_2^2$ and $a = 1 + x_2^2$. Calculate
(a) ∇u.
(b) Δu.
(c) $\nabla \cdot a\nabla u$.
Solution:
(a)

$$\nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}\right) = (x_2^2,\, 2x_1x_2)$$

(b)

$$\Delta u = \nabla\cdot\nabla u = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)\cdot(x_2^2,\, 2x_1x_2) = 0 + 2x_1 = 2x_1$$

(c)

$$\begin{aligned}
\nabla\cdot(a\nabla u) &= \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)\cdot(1+x_2^2)(x_2^2,\, 2x_1x_2) \\
&= \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)\cdot(x_2^2 + x_2^4,\, 2x_1x_2 + 2x_1x_2^3)
\end{aligned}$$

$$= \quad 2x_1 + 6x_1x_2^2$$

□

Problem 10. Consider the problem: find u such that

$$-\Delta u + cu = f \qquad \text{in } \Omega, \tag{27}$$

$$u = g_D \qquad \text{on } \Gamma_D, \tag{28}$$

$$-n \cdot \nabla u = g_N \qquad \text{on } \Gamma_N, \tag{29}$$

where $c = c(x,\, y) \geq 0$, with the usual notation.
(a) Derive a finite element method for this problem using approximation of the Dirichlet boundary condition.
(b) Prove that the finite element solution is unique when 1. $c > 0$ and 2. Γ_D is non-empty.
Solution:
(a) We approximate the Dirichlet boundary condition (28) by

$$-n \cdot \nabla u = \gamma(u - g_D) \quad \text{on } \Gamma_D, \tag{30}$$

where $\gamma >> 0$.
Multiply the differential equation (27) by a function $v = v(x,\, y)$ and integrate over Ω:

$$-\iint_\Omega (\Delta u) v \, dxdy + \iint_\Omega cuv \, dxdy = \iint_\Omega fv \, dxdy.$$

Integrate by parts in the first term:

$$-\int_\Gamma (n \cdot \nabla u) v \, ds + \iint_\Omega \nabla u \cdot \nabla v \, dxdy + \iint_\Omega cuv \, dxdy = \iint_\Omega fv \, dxdy.$$

Use the boundary conditions (29) and (30) to replace $-(n \cdot \nabla u)$ in the boundary integral:

$$\int_{\Gamma_N} g_N v \, ds + \int_{\Gamma_D} \gamma(u - g_D) v \, ds + \iint_\Omega \nabla u \cdot \nabla v \, dxdy + \iint_\Omega cuv \, dxdy = \iint_\Omega fv \, dxdy.$$

We now state the *variational formulation*: Find $u \in V$ such that

$$\begin{aligned} &\int_{\Gamma_D} \gamma uv \, ds + \iint_\Omega \nabla u \cdot \nabla v \, dxdy + \iint_\Omega cuv \, dxdy = \\ &\int_{\Gamma_D} \gamma g_D v \, ds - \int_{\Gamma_N} g_N v \, ds + \iint_\Omega fv \, dxdy \quad \text{for all } v \in V, \end{aligned} \tag{31}$$

where V is the space of functions that are smooth enough for the integrals in (31) to exist.
The corresponding *Finite Element Method* reads: Find $U \in V_h$ such that

$$\int_{\Gamma_D} \gamma Uv \, ds + \iint_\Omega \nabla U \cdot \nabla v \, dxdy + \iint_\Omega cUv \, dxdy =$$

$$\int_{\Gamma_D} \gamma g_D v \, ds - \int_{\Gamma_N} g_N v \, ds + \iint_\Omega f v \, dxdy \quad \text{for all } v \in V_h, \tag{32}$$

where V_h is the space of continuous, piece-wise linear functions on a given triangulation of Ω.

(b) Assume that there are *two* solutions U_1, $U_2 \in V_h$ to (32):

$$\int_{\Gamma_D} \gamma U_1 v \, ds + \iint_\Omega \nabla U_1 \cdot \nabla v \, dxdy + \iint_\Omega c U_1 v \, dxdy =$$
$$\int_{\Gamma_D} \gamma g_D v \, ds - \int_{\Gamma_N} g_N v \, ds + \iint_\Omega f v \, dxdy \quad \text{for all } v \in V_h,$$

$$\int_{\Gamma_D} \gamma U_2 v \, ds + \iint_\Omega \nabla U_2 \cdot \nabla v \, dxdy + \iint_\Omega c U_2 v \, dxdy =$$
$$\int_{\Gamma_D} \gamma g_D v \, ds - \int_{\Gamma_N} g_N v \, ds + \iint_\Omega f v \, dxdy \quad \text{for all } v \in V_h.$$

Subtraction gives:

$$\int_{\Gamma_D} \gamma(U_1 - U_2) v \, ds + \iint_\Omega \nabla(U_1 - U_2) \cdot \nabla v \, dxdy + \iint_\Omega c(U_1 - U_2) v \, dxdy = 0,$$

for all $v \in V_h$. Now choose $v = U_1 - U_2 \in V_h$:

$$\int_{\Gamma_D} \gamma(U_1 - U_2)^2 \, ds + \iint_\Omega |\nabla(U_1 - U_2)|^2 \, dxdy + \iint_\Omega c(U_1 - U_2)^2 \, dxdy = 0. \tag{33}$$

Since all three terms on the left-hand side are non-negative they must all be equal to 0:

$$\int_{\Gamma_D} \gamma(U_1 - U_2)^2 \, ds = 0, \tag{34}$$

$$\iint_\Omega |\nabla(U_1 - U_2)|^2 \, dxdy = 0, \tag{35}$$

$$\iint_\Omega c(U_1 - U_2)^2 \, dxdy = 0. \tag{36}$$

We now consider the two cases separately:

1. If $c > 0$ equation (36) immediately implies that $U_1 - U_2 = 0$ in Ω, i.e., $U_1 = U_2$ in Ω.
2. If we only know that $c \geq 0$, but Γ_D is non-empty, we can first use (35) to conclude that $|\nabla(U_1 - U_2)| = 0$ in Ω, i.e., $U_1 - U_2$ is *constant* in Ω. Then we use (34) to conclude that $U_1 - U_2 = 0$ on Γ_D, but then the constant must be 0 and we have that $U_1 - U_2 = 0$ in Ω, i.e., $U_1 = U_2$ in Ω.

Remark. Since existence and uniqueness is equivalent for quadratic linear systems of equations, we have also proved existence of a solution to our Finite Element Method.

□

Problem 11. Let K be a triangle with corners $(0,0)$, $(0,1)$, and $(1,0)$, and let $f(x_1, x_2) = x_1^2 + x_2$. Calculate

$$\iint_K f(x_1, x_2)\, dx_1 dx_2,$$

using
(a) one-point ("center of mass") quadrature,
(b) corner ("node") quadrature,
(c) mid-point (of the triangle sides) quadrature.
Also compute the integral analytically and compare with your results above.
Solution: Denote the area of K by $\mu(K)$, i.e., $\mu(K) = \frac{1}{2}$.
(a) The co-ordinates for the *center of mass* of a triangle, (x_{CM}, y_{CM}), are the mean values of the co-ordinates of the corners:

$$(x_{CM}, y_{CM}) = \frac{(0,\,0) + (0,\,1) + (1,\,0)}{3} = (\frac{1}{3}, \frac{1}{3}).$$

Thus:

$$\iint_K f(x_1, x_2)\, dx_1 dx_2 \approx f(x_{CM}, y_{CM})\, \mu(K) = \left(\left(\frac{1}{3}\right)^2 + \frac{1}{3}\right) \cdot \frac{1}{2} = \frac{2}{9}.$$

(b)

$$\iint_K f(x_1, x_2)\, dx_1 dx_2 \approx \frac{f(0,\,0) + f(0,\,1) + f(1,\,0)}{3}\, \mu(K) = \frac{0+1+1}{3} \cdot \frac{1}{2} = \frac{1}{3}.$$

(c)

$$\begin{aligned}\iint_K f(x_1, x_2)\, dx_1 dx_2 &\approx \frac{f(0,\,1/2) + f(1/2,\,0) + f(1/2,\,1/2)}{3}\, \mu(K) \\ &= \frac{1/2 + 1/4 + 3/4}{3} \cdot \frac{1}{2} = \frac{1}{4}.\end{aligned}$$

We know that the quadrature rule in (c) should give the exact result in this case, since f is a polynomial of degree 2. We check:

$$\iint_K f(x_1, x_2)\, dx_1 dx_2 = \int_0^1 \left(\int_0^{1-x_1} (x_1^2 + x_2)\, dx_2\right) dx_1$$

$$= \int_0^1 \left[x_1^2 x_2 + \frac{1}{2} x_2^2\right]_{x_2=0}^{x_2=1-x_1} dx_1 = \int_0^1 \left(x_1^2(1-x_1) + \frac{1}{2}(1-x_1)^2\right) dx_1$$

$$= \int_0^1 \left(\frac{1}{2} - x_1 + \frac{3}{2}x_1^2 - x_1^3\right) dx_1 = \left[\frac{1}{2}x_1 - \frac{1}{2}x_1^2 + \frac{1}{2}x_1^3 - \frac{1}{4}x_1^4\right]_{x_1=0}^{x_1=1} = \frac{1}{2} - \frac{1}{2} + \frac{1}{2} - \frac{1}{4} = \frac{1}{4}!$$

□

Problem 12. Let K be a triangle with corners $(0,0)$, $(0,1)$, and $(1,0)$.
(a) Calculate the three basis functions λ_i, $i = 1,2,3$, for the space $\mathcal{P}(K)$ of linear functions defined on K.
(b) Calculate the 3×3 element mass matrix with elements $m_{ij} = \iint_K \lambda_j \lambda_i \, dx\, dy$ approximately using corner quadrature.
(c) Calculate the 3×3 element stiffness matrix with elements $a_{ij} = \iint_K \nabla\lambda_j \cdot \nabla\lambda_i \, dx\, dy$.
Solution: Denote the area of K by $\mu(K)$, i.e., $\mu(K) = \frac{1}{2}$. We also introduce the node numbering $N_1 = (0,0)$, $N_2 = (0,1)$, and $N_3 = (1,0)$.
(a) You can compute the basis functions in the same way as you did in *Problem 2(a) (Week 5)*. An alternative is to argue as follows: The basis function $\lambda_3(x, y)$ is equal to 1 in $(1,0)$ and is equal to 0 for $x = 0$. It therefore has to be $\lambda_3(x, y) = x$, since this is a linear function that obviously satisfies these two requirements. (And linear functions are uniquely determined by their nodal values.) By the same argument we have $\lambda_2(x, y) = y$. Finally we know that $\lambda_1(x, y) + \lambda_2(x, y) + \lambda_3(x, y) \equiv 1$ since the sum is a linear function that is equal to 1 in all three nodes. Therefore $\lambda_1(x, y) = 1 - \lambda_3(x, y) - \lambda_2(x, y) = 1 - x - y$.
(b) With corner (node) quadrature we approximate:

$$m_{ij} = \iint_K \lambda_j \lambda_i \, dx\, dy \approx \frac{\lambda_j(0,\,0)\lambda_i(0,\,0) + \lambda_j(0,\,1)\lambda_i(0,\,1) + \lambda_j(1,\,0)\lambda_i(1,\,0)}{3}\,\mu(K).$$

If $i \neq j$ at least one of the factors λ_j and λ_i is zero in each corner and therefore $m_{ij} = 0$. If $i = j$ we get:

$$\begin{aligned} m_{ii} &= \iint_K \lambda_i \lambda_i \, dx\, dy \approx \frac{\lambda_i(0,\,0)\lambda_i(0,\,0) + \lambda_i(0,\,1)\lambda_i(0,\,1) + \lambda_i(1,\,0)\lambda_i(1,\,0)}{3}\,\mu(K) \\ &= \frac{1^2}{3} \cdot \frac{1}{2} = \frac{1}{6}, \end{aligned}$$

since λ_i is equal to 1 in one node and equal to 0 in the other two nodes.

The final result is therefore:

$$\begin{bmatrix} 1/6 & 0 & 0 \\ 0 & 1/6 & 0 \\ 0 & 0 & 1/6 \end{bmatrix}$$

(c) Since the gradient of a linear function is constant we can move the integrand outside the integral:

$$a_{ij} = \iint_K \nabla\lambda_j \cdot \nabla\lambda_i \, dx\, dy = (\nabla\lambda_j \cdot \nabla\lambda_i) \underbrace{\iint_K dx\, dy}_{\mu(K)} = \frac{1}{2}\left(\nabla\lambda_j \cdot \nabla\lambda_i\right).$$

From (a) we can compute: $\nabla\lambda_1 = (-1, -1)$, $\nabla\lambda_2 = (0, 1)$ and $\nabla\lambda_3 = (1, 0)$. We thus get: $\nabla\lambda_1 \cdot \nabla\lambda_1 = 2$, $\nabla\lambda_2 \cdot \nabla\lambda_2 = 1$, $\nabla\lambda_3 \cdot \nabla\lambda_3 = 1$, $\nabla\lambda_1 \cdot \nabla\lambda_2 = \nabla\lambda_2 \cdot \nabla\lambda_1 = -1$, $\nabla\lambda_1 \cdot \nabla\lambda_3 = \nabla\lambda_3 \cdot \nabla\lambda_1 = -1$ and $\nabla\lambda_2 \cdot \nabla\lambda_3 = \nabla\lambda_3 \cdot \nabla\lambda_2 = 0$.

The final result is therefore:

$$\begin{bmatrix} 1 & -1/2 & -1/2 \\ -1/2 & 1/2 & 0 \\ -1/2 & 0 & 1/2 \end{bmatrix}$$

□