94 Taylor's formula

The following is taken from AMBS Ch 28.15 and Problem 28.11.

Theorem 94.1. (Taylor's formula) *Suppose that f has Lipschitz continuous derivative of order $n+1$ on the interval $[a,b]$ and let $\bar{x} \in (a,b)$. Then f satisfies* Taylor's formula of order n at $\bar{x}$*:*

$$\begin{aligned} f(x) &= f(\bar{x}) + f'(\bar{x})(x-\bar{x}) + f''(\bar{x})\frac{(x-\bar{x})^2}{2} + f'''(\bar{x})\frac{(x-\bar{x})^3}{3!} \\ &\quad + \cdots + f^{(n)}(\bar{x})\frac{(x-\bar{x})^n}{n!} + R_n(x,\bar{x}) \\ &= \sum_{k=0}^{n} f^{(k)}(\bar{x})\frac{(x-\bar{x})^k}{k!} + R_n(x), \quad \textit{for all } x \in [a,b], \end{aligned}$$

where the remainder R_n *is given by*

$$R_n(x) = \int_{\bar{x}}^{x} \frac{(x-y)^n}{n!} f^{(n+1)}(y)\,dy \tag{94.1}$$

$$= f^{(n+1)}(\hat{x})\frac{(x-\bar{x})^{n+1}}{(n+1)!}, \tag{94.2}$$

and $\hat{x}$ is an unknown number between x and $\bar{x}$.

The polynomial

$$P_n(x) = \sum_{k=0}^{n} f^{(k)}(\bar{x})\frac{(x-\bar{x})^k}{k!}$$

is called *the Taylor polynomial of f of degree n at $\bar{x}$.* Remember that n *factorial* ("n fakultet") means

$$n! = 1 \cdot 2 \cdot 3 \cdots n, \quad 0! = 1.$$

The Taylor formula expresses the function as a sum of two terms:

$$f(x) = P_n(x) + R_n(x),$$

where the polynomial is simple to compute with, and where the remainder has a more complicated, perhaps unknown, dependence on x. Note that both the Taylor polynomial and the remainder also depend on $\bar{x}$ but we always consider $\bar{x}$ as a fixed number. Therefore we write $R_n(x)$ instead of $R_n(x,\bar{x})$.

The main importance of the formula is that it gives a formula for the remainder, which shows that the remainder is smaller than the terms in the polynomial, when x is close to $\bar{x}$. For example, if we know that

$$|f^{(n+1)}(x)| \le M, \quad \text{for all } x \in [a,b],$$

then, by (94.2),

$$|R_n(x)| = |f^{(n+1)}(\hat{x})|\,\frac{|x-\bar{x}|^{n+1}}{(n+1)!} \le M\frac{|x-\bar{x}|^{n+1}}{(n+1)!}. \tag{94.3}$$

This means that we can often compute with the Taylor polynomial, which is simple to do, and then draw conclusions about f. This also means that we can write

$$f(x) = P_n(x) + B_n(x)(x-\bar{x})^{n+1} \tag{94.4}$$

where the function B_n is bounded near $\bar{x}$:

$$|B_n(x)| = |f^{(n+1)}(\hat{x})|\,\frac{1}{(n+1)!} \le \frac{M}{(n+1)!}.$$

This form of Taylor's formula makes it easy compute limits as $x \to \bar{x}$, see Problem 94.3.

Note the two forms of the remainder, (94.1) and (94.2). The second one is often easier to use. Note also that we need not know $\hat{x}$, it is usually sufficient to have a rough estimate of $f^{(n+1)}(\hat{x})$ as in (94.3).

Proof. We begin by recalling from the Fundamental Theorem of Calculus:

$$f(x) = f(\bar{x}) + \int_{\bar{x}}^{x} f'(y)\, dy$$

Note that this is Taylor's formula of order 0: $P_0(x) = f(\bar{x})$ is Taylor's polynomial of degree 0 with remainder $R_0(x) = \int_{\bar{x}}^{x} f'(y)\, dy$. To obtain Taylor's formula of order 1 we integrate by parts:

$$\begin{aligned} f(x) &= f(\bar{x}) + \int_{\bar{x}}^{x} 1 \cdot f'(y)\, dy \\ &= f(\bar{x}) + \int_{\bar{x}}^{x} \frac{d}{dy}(y-x) f'(y)\, dy \\ &= f(\bar{x}) + \Big[(y-x)f'(y)\Big]_{\bar{x}}^{x} - \int_{\bar{x}}^{x} (y-x) f''(y)\, dy \\ &= \underbrace{f(\bar{x}) + f'(\bar{x})(x-\bar{x})}_{=P_1(x)} + \underbrace{\int_{\bar{x}}^{x} (x-y) f''(y)\, dy}_{=R_1(x)}. \end{aligned}$$

Note that this is the same as our definition of the derivative:

$$f(x) = f(\bar{x}) + f'(\bar{x})(x-\bar{x}) + E_f(x,\bar{x}).$$

We continue and integrate by parts once more using $(x-y) = -\frac{d}{dy}\frac{(x-y)^2}{2}$:

$$\begin{aligned} f(x) &= f(\bar{x}) + f'(\bar{x})(x-\bar{x}) - \int_{\bar{x}}^{x} \frac{d}{dy}\frac{(x-y)^2}{2} f''(y)\, dy \\ &= f(\bar{x}) + f'(\bar{x})(x-\bar{x}) - \Big[\frac{(x-y)^2}{2} f''(y)\Big]_{\bar{x}}^{x} + \int_{\bar{x}}^{x} \frac{(x-y)^2}{2} f'''(y)\, dy \\ &= f(\bar{x}) + f'(\bar{x})(x-\bar{x}) + f''(\bar{x})\frac{(x-\bar{x})^2}{2} + \int_{\bar{x}}^{x} \frac{(x-y)^2}{2} f'''(y)\, dy \\ &= P_2(x) + R_2(x). \end{aligned}$$

Repeating this once more we get:

$$\begin{aligned} f(x) &= f(\bar{x}) + f'(\bar{x})(x-\bar{x}) + f''(\bar{x})\frac{(x-\bar{x})^2}{2} + f'''(\bar{x})\frac{(x-\bar{x})^3}{2\cdot 3} + \int_{\bar{x}}^{x} \frac{(x-y)^3}{2\cdot 3} f^{(4)}(y)\, dy \\ &= P_3(x) + R_3(x). \end{aligned}$$

After n steps of this procedure we have:

$$\begin{aligned} f(x) &= f(\bar{x}) + f'(\bar{x})(x-\bar{x}) + f''(\bar{x})\frac{(x-\bar{x})^2}{2} + f'''(\bar{x})\frac{(x-\bar{x})^3}{3!} + \cdots + f^{(n)}(\bar{x})\frac{(x-\bar{x})^n}{n!} \\ &\quad + \int_{\bar{x}}^{x} \frac{(x-y)^n}{n!} f^{(n+1)}(y)\, dy = P_n(x) + R_n(x). \end{aligned}$$

This is Taylor's formula of order n with remainder in the form (94.1).

In order to obtain the alternative form (94.2) we first make a transformation of variable:

$$R_n(x) = \int_{\bar{x}}^{x} \frac{(x-y)^n}{n!} f^{(n+1)}(y)\,dy = \left\{ \begin{array}{l} z = \dfrac{(x-y)^{n+1}}{(n+1)!} \\ dz = -\dfrac{(x-y)^n}{n!}\,dy \\ y = \bar{x} \implies z = \bar{z} = \dfrac{(x-\bar{x})^{n+1}}{(n+1)!} \\ y = x \implies z = 0 \end{array} \right\}$$

$$= \int_0^{\bar{z}} f^{(n+1)}(y(z))\,dz,$$

and then use the mean value theorem for integrals, which says that there is an unknown number $\hat{z}$ between 0 and $\bar{z}$ such that

$$R_n(x) = \int_0^{\bar{z}} f^{(n+1)}(y(z))\,dz = f^{(n+1)}(y(\hat{z})) \int_0^{\bar{z}} dz = f^{(n+1)}(y(\hat{z}))\bar{z} = f^{(n+1)}(\hat{x}) \frac{(x-\bar{x})^{n+1}}{(n+1)!}$$

where $\hat{x} = y(\hat{z})$, that is, $\hat{z} = \frac{(x-\hat{x})^{n+1}}{(n+1)!}$. □

Problems

94.1. Write down Taylor's formula of order n at $\bar{x} = 0$ for the following functions:

(a) $\log(1+x)$

(b) $\exp(x)$

(c) $\sin(x)$

(d) $\cos(x)$

94.2. Use Taylor's formula of order 2 (or 3 or 4) to compute approximations of the following. Estimate the error.

(a) $\log(1.1)$

(b) $\exp(-0.1)$

(c) $\sin(0.1)$

(d) $\cos(0.1)$

94.3. Compute the following limits.

(a) $\displaystyle\lim_{x\to 0} \frac{\sin(x)}{x}$

(b) $\displaystyle\lim_{x\to 0} \frac{1-\cos(x)}{x^2}$

(c) $\displaystyle\lim_{x\to 0} \frac{e^x - 1 - \sin(x)}{\log(1+x) - x}$

(d) $\displaystyle\lim_{x\to 0} \frac{\log(1+x^2) - x^2}{x^4}$

Answers and solutions

94.1.

(a)

$$
\begin{aligned}
f(x) &= \log(1+x) & f(0) &= 0\\
f'(x) &= \frac{1}{1+x} & f'(0) &= 1\\
f''(x) &= \frac{-1}{(1+x)^2} & f''(0) &= -1\\
f'''(x) &= \frac{(-1)(-2)}{(1+x)^3} = \frac{(-1)^2 2!}{(1+x)^3} & f'''(0) &= 2 = (-1)^2 2!\\
f''''(x) &= \frac{(-1)(-2)(-3)}{(1+x)^4} = \frac{(-1)^3 3!}{(1+x)^4} & f''''(0) &= -6 = (-1)^3 3!\\
&\vdots & &\vdots\\
f^{(k)}(x) &= \frac{(-1)^{k-1}(k-1)!}{(1+x)^k} & f^{(k)}(0) &= (-1)^{k-1}(k-1)!
\end{aligned}
$$

$$
\begin{aligned}
\log(1+x) &= 0 + x + (-1)\frac{x^2}{2!} + (-1)^2 2!\frac{x^3}{3!} + (-1)^3 3!\frac{x^4}{4!} + \cdots + (-1)^{n-1}(n-1)!\frac{x^n}{n!} + R_n(x,0)\\
&= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n-1}\frac{x^n}{n} + R_n(x,0)\\
&= \sum_{k=0}^{n}(-1)^{k-1}\frac{x^k}{k} + R_n(x,0)\\
R_n(x,0) &= \frac{(-1)^n n!}{(1+\hat{x})^{n+1}}\frac{x^{n+1}}{(n+1)!} = \frac{(-1)^n}{(1+\hat{x})^{n+1}}\frac{x^{n+1}}{n+1}, \quad \text{where } \hat{x} \text{ is between } x \text{ and } 0.
\end{aligned}
$$

(b)

$$
\begin{aligned}
\exp(x) &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} + R_n(x,0)\\
R_n(x,0) &= e^{\hat{x}}\frac{x^{n+1}}{(n+1)!}, \quad \text{where } \hat{x} \text{ is between } x \text{ and } 0
\end{aligned}
$$

(c)

$$
\begin{aligned}
f(x) &= \sin(x) & &(k=0,\ m=1) & f(0) &= 0\\
f'(x) &= \cos(x) & &(k=1,\ m=1) & f'(0) &= 1\\
f''(x) &= -\sin(x) & &(k=2,\ m=2) & f''(0) &= 0\\
f'''(x) &= -\cos(x) & &(k=3,\ m=2) & f'''(0) &= -1\\
f''''(x) &= \sin(x) & &(k=4,\ m=3) & f''''(0) &= 0\\
f^{(5)}(x) &= \cos(x) & &(k=5,\ m=3) & f^{(5)}(0) &= 1\\
&\vdots & &\vdots & &\vdots\\
f^{(2m-2)}(x) &= (-1)^{m-1}\sin(x) & &(k=2m-2 \text{ even}) & f^{(2m-2)}(0) &= 0\\
f^{(2m-1)}(x) &= (-1)^{m-1}\cos(x) & &(k=2m-1 \text{ odd}) & f^{(2m-1)}(0) &= (-1)^{m-1}
\end{aligned}
$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + R_{2n}(x,0)$$
$$R_{2n}(x,0) = (-1)^n \cos(\hat{x})\frac{x^{2n+1}}{(2n+1)!}, \quad \text{where } \hat{x} \text{ is between } x \text{ and } 0$$

(d)

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n\frac{x^{2n}}{(2n)!} + R_{2n+1}(x,0)$$
$$R_{2n+1}(x,0) = (-1)^{n+1} \cos(\hat{x})\frac{x^{2n+2}}{(2n+2)!}, \quad \text{where } \hat{x} \text{ is between } x \text{ and } 0$$

94.2.

(a) Taylor of order 2:

$$\log(1+x) = x - \frac{x^2}{2} + R_2(x,0)$$
$$R_2(x,0) = \frac{1}{(1+\hat{x})^3}\frac{x^3}{3}$$
$$\log(1.1) = \log(1+0.1) \approx 0.1 - \frac{(0.1)^2}{2} = 0.1 - 0.005 = 0.095$$
$$|R_2(0.1,0)| = \left|\frac{1}{(1+\hat{x})^3}\frac{(0.1)^3}{3}\right| = \frac{1}{(1+\hat{x})^3}\frac{(0.1)^3}{3} \le \frac{1}{3}\cdot 10^{-3}$$

because $\hat{x} \in [0, 0.1]$ implies $1+\hat{x} \ge 1$, so that $\frac{1}{(1+\hat{x})^3} \le 1$. Thus, $\log(1.1) \approx 0.095$ with 3 correct decimals.

(b) Taylor of order 2:

$$\exp(x) = 1 + x + \frac{x^2}{2} + R_2(x,0)$$
$$R_2(x,0) = e^{\hat{x}}\frac{x^3}{3!}$$
$$\exp(-0.1) \approx 1 + (-0.1) + \frac{(-0.1)^2}{2} = 0.905$$
$$|R_2(-0.1,0)| = \left|e^{\hat{x}}\frac{(-0.1)^3}{3!}\right| = \frac{1}{6}e^{\hat{x}}10^{-3} \le \frac{1}{6}\cdot 10^{-3}$$

because $\hat{x} \in [-0.1, 0]$ implies $e^{\hat{x}} \le 1$. Thus, $\exp(-0.1) \approx 0.905$ with 3 correct decimals.

(c) Taylor of order 4:

$$\sin(x) = x - \frac{x^3}{6} + R_4(x,0)$$
$$R_4(x,0) = (-1)^2\cos(\hat{x})\frac{x^5}{5!}$$
$$\sin(0.1) \approx 0.1 - \frac{(0.1)^3}{6} \approx 0.099833333$$
$$|R_4(0.1,0)| = \left|(-1)^2\cos(\hat{x})\frac{(0.1)^5}{5!}\right| = |\cos(\hat{x})|\frac{1}{120}10^{-5} \le \frac{1}{120}\cdot 10^{-5} < 10^{-7}$$

because $|\cos(\hat{x})| \le 1$. Thus, $\sin(0.1) \approx 0.099833$ with 6 correct decimals.

(d) ...

94.3.

(a) By using (94.4) and Problem 94.1(c) we get

$$\lim_{x\to 0}\frac{\sin(x)}{x}=\lim_{x\to 0}\frac{x+B_3(x)x^3}{x}=\lim_{x\to 0}(1+B_3(x)x^2)=1$$

Note that $B_3(x)=-\cos(\hat{x})/3$ is bounded near $x=0$ because $|B_3(x)|\leq 1/3$.

(b) 1

(c) -1

(d) -1/2

2005-11-27 /stig