Answers to the Problems in EM-2000

October 31, 2000

For errors in the answers or statements of the problems, please send email to Mats Larson: `mgl@math.chalmers.se`. Updated versions of this document will be available on: `http://www.md.chalmers.se/Centres/Phi/Education/KfKb/Kurser/AnalysA/`.

Chapter 3

3.2 $17x = 10$

3.5 $x^2 = 3$

3.6 $170x = 45(12 - x)$

3.8 $x = 23/33, y = -5/33$ (corresponding to *selling* some ice cream to get more money to spend on soup!)

Chapter 5

5.1 1) My age, 2) Number of my children, 3) Number of contries that I have seen, 4) Number of languages that I speak, 5) Number of Vivaldi music CD that I own

5.5 In $m \times n = 0$ if and only if $m = 0$ *or* $n = 0$, *or* means *either, or* i.e. either $m = 0$ or $n = 0$.

5.8 (a) $102 = 5 \times 18 + 12$

(b) $-4301 = -69 \times 63 + 46$

(c) $650912 = 2106 \times 309 + 158$

5.9 (a) $40 = 2^3 \times 5 \Longrightarrow \{1, 2, 4, 8, 10, 20, 40\}|40$

(b) $80 = 2^4 \times 5 \Longrightarrow \{1, 2, 4, 8, 10, 16, 20, 40, 80\}|80$

5.10 For what *or* means, see 5.5.

If in $p \times m = p \times n,\ p = 0$, then m and n could be any nonzero (because 0×0 is NOT defined) integer number, (for example $m = 17, n = -1$).

5.12 $(a+b)^2 = a^2 + b^2$ is not valid. Simply take $a = b = 1$, then the left hand side is 4 while the right hand side is 2.

$ac < bc$ implies $a < b$ is an invalid implication. Just take $a = 2, b = 1, c = -1$, then we are getting: $-2 < -1$ implies $2 < 1$, i.e., we derive from a correct statement a wrong conclusion.

Finally $a + bc = (a+b)c$. Take, for example, $a = b = 1$ and $c = 0$, you get $1 = 0$.

5.13 (a) $-2 < x < 20$

(b) $8 < x < 20$

(c) $-13 < x < 25$

(d) $1 < x < 3$

Chapter 6

6.1 (a) The inductive step: $1^2 + 2^2 + 3^2 + \ldots + (n-1)^2 + n^2 =$ (the inductive assumption) $= \frac{(n-1)n(2(n-1)+1)}{6} + n^2 = \frac{2n^3-3n^2+n+6n^2}{6} = \frac{n(n+1)(2n+1)}{6}$

(b) The inductive step: $1^3 + 2^3 + 3^3 + \ldots + (n-1)^3 + n^3 =$ (the inductive assumption) $= (\frac{(n-1)n}{2})^2 + n^3 = \frac{n^4-2n^3+n^2+4n^3}{4} = (\frac{n(n+1)}{2})^2$

6.2 Note: error in the problem statement $1/(n+1)$ should be $n/(n+1)$. The inductive step: $\frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \ldots + \frac{1}{(n-1)n} + \frac{1}{n(n+1)} =$ (the inductive assumption) $= \frac{n-1}{(n-1)+1} + \frac{1}{n(n+1)} = \frac{(n-1)(n+1)+1}{n(n+1)} = \frac{n}{n+1}$

6.3 (a) The inductive step: $3n^2 = 3((n-1)+1)^2 = 3(n-1)^2 + 6n - 3 \geq$ (the inductive assumption) $\geq 2(n-1)+1+6n-3 = 2n+(6n-4) \geq 2n+1$

(b) The inductive step: $4^n = 4 \times 4^{n-1} \geq$ (the inductive assumption) $\geq 4(n-1)^2 = n^2 + (3n^2 - 8n + 4) = n^2 + 3(n-2)^2 + 4(n-2) \geq n^2$ (for $n \geq 2$)

6.4 Let P_n denote the size of the population year n. The modeling assumption is that $P_n = KP_{n-1}^2$, which iterated n times gives $P_n = K^{2^n-1}P_0^{2^n}$.

6.5 Let P_n denote the size of the population year n. The modeling assumption is that $P_n = K_1P_{n-1} - K_2P_{n-1}^2$.

6.6 Let P_n denote the size of the population year n. The modeling assumption is that $P_n = KP_{n-1} + KP_{n-2}$.

6.7 –

6.8 The inductive step: Since, by long division, $\begin{array}{r|lll} & & p^n & \\ \hline p-1 & p^{n+1} & & -1 \\ & -(\; p^{n+1} & -p^n \;) & \\ \hline & & p^n & -1 \end{array}$

we get $\frac{p^{n+1}-1}{p-1} = p^n + \frac{p^n-1}{p-1} =$ (the inductive assumption) $= p^n + p^{n-1} + \ldots + 1$

Chapter 7

7.3 Proof of Commutative law for addition:
$(p,q)+(r,s) = (ps,qs)+(qr,qs) = (ps+qr,qs) = (rq+sp,sq) = (rq,sq)+(sp,sq) = (r,s)+(p,q)$
Proof of Commutative law for multiplication:
$(p,q)\times(r,s) = (pr,qs) = (rp,sq) = (r,s)\times(p,q)$
Proof of Distributive law:
$(p,q)\times((r,s)+(t,u)) = (p,q)\times(ru+st,su) = (p(ru+st),qsu) = pru+pst,qsu) = (pru,qsu)+(pst,qsu) = (pr,qs)+(pt,qu) = (p,q)\times(r,s)+(p,q)\times(t,u)$

7.4 For rational numbers $r = \frac{r_1}{r_2}$, $s = \frac{s_1}{s_2}$ and $t = \frac{t_1}{t_2}$, one has

$$r(s+t) = \frac{r_1}{r_2}\left(\frac{s_1}{s_2} + \frac{t_1}{t_2}\right) = \frac{r_1}{r_2}\,\frac{s_1t_2+s_2t_1}{s_2t_2} = \frac{r_1s_1t_2+r_1s_2t_1}{r_2s_2t_2}$$

$$= \frac{r_1s_1t_2}{r_2s_2t_2} + \frac{r_1s_2t_1}{r_2s_2t_2} = \frac{r_1s_1}{r_2s_2} + \frac{r_1t_1}{r_2t_2} = rs+rt$$

7.5 (a) $\{x \in Q : 1 \le x \le 5/3\}$
(b) $\{x \in Q : -\frac{4}{5} < x < \frac{8}{5}\}$
(c) $\{x \in Q : x < -\frac{1}{14}$ or $x > \frac{13}{14}\}$
(d) $\{x \in Q : x \le -\frac{1}{8}$ or $x \ge \frac{5}{8}\}$

7.6 Using the fact that one mile is 5280 feet, and one hour 3600 seconds, the speed of the runner is 16 miles/hour plus 8.8 feet/second, that is $16 \times 5280/3600 + 8.8 = \frac{16\cdot5280}{3600} + \frac{8.8\cdot3600}{3600} = \frac{84480+31680}{3600} = \frac{116160}{3600}$ feet/second, that is 32.26666.. feet/second.

7.7 (a) 0.428571 428571 42..
(b) 0.153846 153846 15..
(c) 0.2941176470588235 29411..

7.8 (a) 3.456
(b) 0.5975

7.9 (a) 42/99, that is 14/13

(b) 8811/9999

(c) 42905/99999

7.10 (a) Skippas!

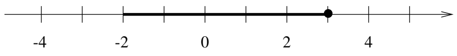
(b) 

Figure 1:

(c) Skippas!

(d) Skippas!

7.11

7.12 $C_0(1+0.09)^n$

Chapter 9

9.1

9.2 (a) $(-10, 14]$

(b) $(10, \infty)$

(c) $(-14.8, 25.2]$

9.3 $(-\infty, 0) \cup [\frac{1}{13}, \frac{3}{13}]$

9.4 domain: $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$,
range: $\{75, 75.01, 75.08, 75.27, 75.64, 76.25, .., 82.29\}$

9.5 $D_H = [0, 50]$, $R_H = [0, \sqrt{50}]$

9.6 $\{1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, ..\}$

9.7 $D_f = Q = (-\infty, \infty)$. B could be any set containing $R_f = (0, 1]$

9.8 (a) $\{x \in Q : x \neq -2 \text{ and } x \neq 4 \text{ and } x \neq 5\}$

(b) $\{x \in Q : x \neq -2 \text{ and } x \neq 2\}$

(c) $\{x \in Q : x \neq -1/2 \text{ and } x \neq 8\}$

9.9 $\{0, 1, 2, 3, 4\}$

9.10 Skippas!

9.11 (a) (b), (c) skippas!

9.12 Skippas!

9.13 Skippas!

9.14 Skippas!

9.15 Skippas!

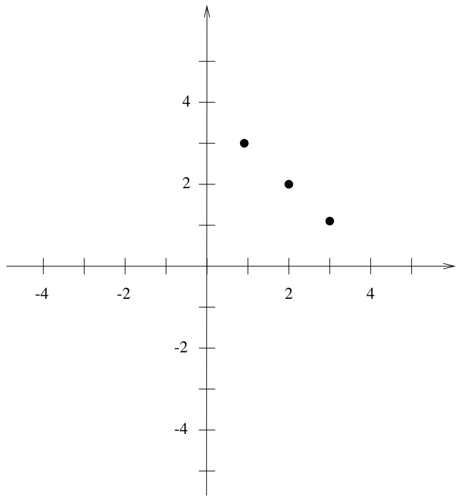


Figure 2:

Chapter 10

10.1 (a) $y = 4x - 1$

(b) $y - 2 = -\frac{1}{2}(x + 4)$

(c) $y - 7 = 0$

(d) $y = -7.4x + 27$

(e) $x = -3$

(f) $y = -3x + 5$

10.2 (a) $y = \frac{4}{5}x - \frac{46}{5}$

(b) $y = -\frac{3}{2}x + \frac{5}{2}$

(c) $x = 13$

(d) $y = 4$

(e) $y = 10x + 7$

(f) $y = -2x - 1$

10.3 $x = -\frac{b}{m}$

10.4 See the plot of the functions below.

10.5 Yes!

10.6 Yes!

10.7 (a) $(-4/7, 2/7)$

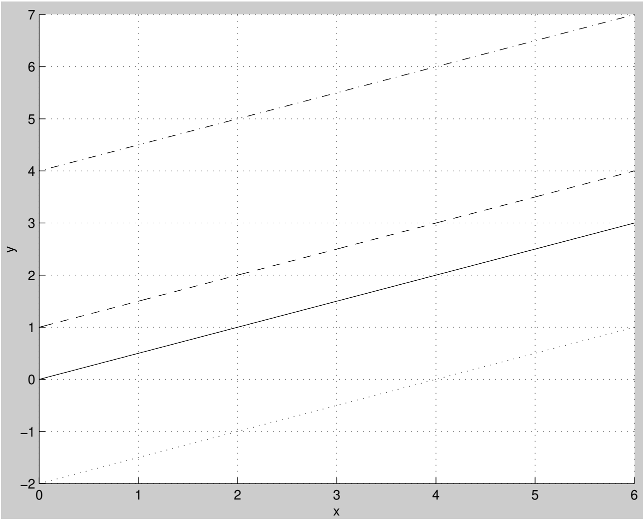


Figure 3: A plot of the functions $y = \frac{1}{2}x$ (___), $y = \frac{1}{2}x - 2$ (...), $y = \frac{1}{2}x + 4$ (_._) and $y = \frac{1}{2}x + 1$ (_ _ _). (Problem 10.4)

(b) $(35/11, 223/11)$

10.8 $y = -\frac{1}{10}(x - 3)$

10.9 (a) $y = 0.1x + 1.9$

(b) $y = -10x + 12$

10.10 $x_1 < x_2 < 0$ implies $x_2^2 - x_1^2 = (x_2 + x_1)(x_2 - x_1) < 0$, because $x_2 + x_1 < 0$ and $x_2 - x_1 > 0$, that is $x_2^2 < x_1^2$.

10.11 See the plots of the functions below.

10.12 See the plots of the functions below.

10.13 See the plots of the functions below.

10.14 See plot below.

10.15 See plot below

(a) $y = x^2 + 4x + 5 = (x + 2)^2 + 1$

(b) $y = 2x^2 - 2x - \frac{1}{2} = 2(x - \frac{1}{2})^2 - 1$

(c) $y = -\frac{1}{3}x^2 + 2x - 1 = -\frac{1}{3}(x - 3)^2 + 2$

10.16 (a) $\sum_{i=1}^{n} i^{-2}$

(b) $\sum_{i=1}^{n} (-1)^i i^{-2}$

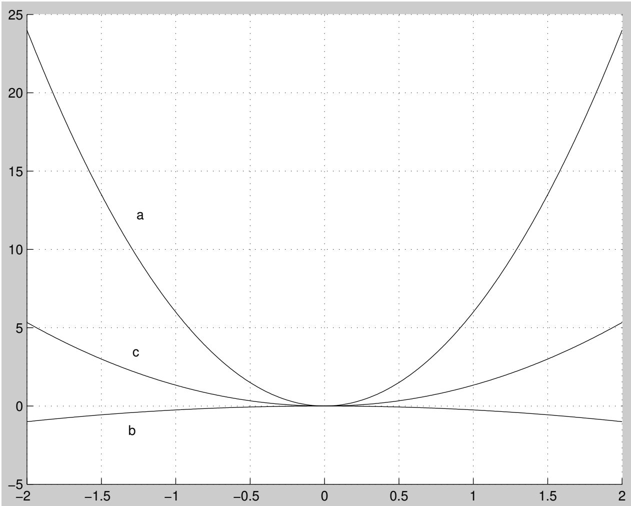


Figure 4: Plot of the functions a) $y = 6x^2$, b) $y = \frac{1}{4}x^2$, and c) $y = \frac{4}{3}x^2$. (Problem 10.11)

(c) $\frac{1}{2} + \sum_{i=1}^{n} \frac{1}{i(i+1)}$

(d) $\sum_{i=0}^{n}(2i+1)$

(e) $\sum_{i=0}^{n-4} x^{4+i}$

(f) $\sum_{i=0}^{n} x^{2i}$

10.17 (a) $\sum_{i=-1}^{n-2}(i+2)^2$

(b) $\sum_{i=15}^{n+14}(i-14)^2$

(c) $\sum_{i=-3}^{n-4}(i+4)^2$

(d) $\sum_{i=8}^{n+7}(i-7)^2$

10.18 (a) $-4 + 6x - 8x^2 + 11x^3 - 16x^5$

(b) $48 - 72x + 6x^2 - 87x^3 + 12x^5$

(c) $-2 + 6x + 2x^2 + 6x^3 - x^4 + 4x^5$

(d) $-8x^2 + 16x^3 - 6x^4 - 2x^5 + 17x^6 + 28x^8$

(e) $-8 + 12x + 14x^3 + 4x^4 - 6x^5 - 7x^7$

(f) $4x^2 - 2x^3 + 8x^5 - 2x^6 + x^7 - 4x^9$

(g) $-8 + 12x - 2x^2 + 15x^3 + 4x^4 - 10x^5 - 7x^7$

(h) $-8 + 12x + 4x^2 + 12x^3 + 4x^4 + 2x^5 - 2x^6 - 6x^7 - 4x^9$

(i) $-16x^2 + 32x^3 - 12x^4 - 4x^5 + 42x^6 - 16x^7 + 62x^8 + 2x^9 - 17x^{10} - 28x^{12}$

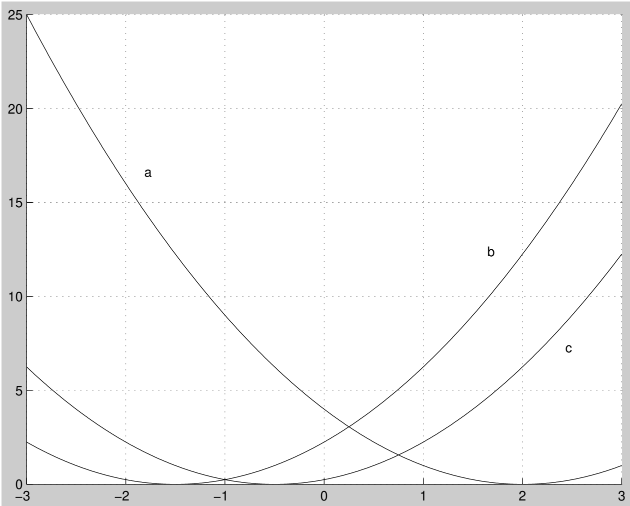


Figure 5: Plots of the functions a) $y = (x-2)^2$, b) $y = (x+1.5)^2$, and c)$y = (x+0.5)^2$. (Problem 10.12a)

10.19 (a) $x^2 + 2xa + a^2$

(b) $x^3 + 3x^2a + 3xa^2 + a^3$

(c) $x^3 - 3x^2a + 3xa^2 - a^3$

(d) $x^4 + 4x^3a + 6x^2a^2 + 4xa^3 + a^4$

10.20 $p_1p_2 = \sum_{i=0}^{8}\sum_{j=0}^{11} \frac{i}{j+1} x^{i+j}$

10.21 The polynomial $p(x) = 360x - 942x^2 + 949x^3 - 480x^4 + 130x^5 - 18x^6 + x^7$ is zero for 0,1,2,3,4,5, and has the property that $p(x) \to +\infty$ when $x \to +\infty$ and $p(x) \to -\infty$ when $x \to -\infty$, see plots below.The polynomial can be factored into $p(x) = x(x-1)(x-2)(x-3)^2(x-4)(x-5)$, which explains the behavior.

10.22 (a) *Has increasing/decreasing been defined in the book?* A function f is increasing in an interval (a, b) if $a < x \leq y < b$ implies $f(x) \leq f(y)$. From $x^3 - y^3 = \frac{1}{2}(x-y)(x^2 + y^2 + (x+y)^2)$ it is seen that $x^3 - y^3$ has the same sign as $x - y$, hence x^3 is increasing.

(b) A function f is decreasing in an interval (a, b) if $a < x \leq y < b$ implies $f(x) \geq f(y)$. From $x^4 - y^4 = (x-y)(x+y)(x^2 + y^2)$ it is seen that $x^4 \leq y^4$ if $0 < x \leq y$, and $x^4 \geq y^4$ if $x \leq y < 0$.

10.23 Reformulation of problem: Plot the monomials for $-2 \leq x \leq 2$. See the plot below.

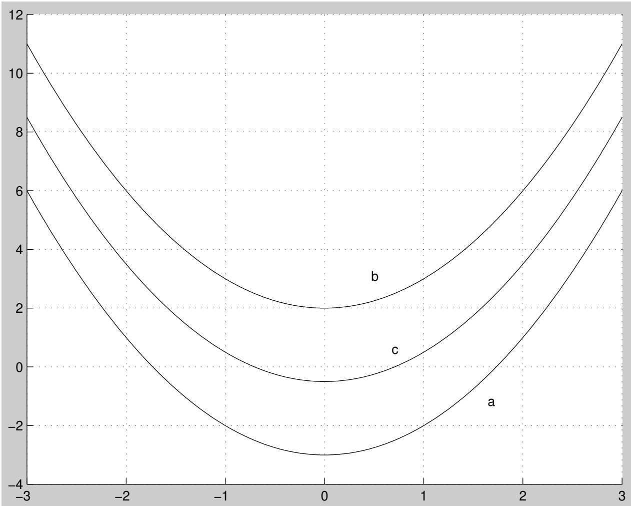


Figure 6: Plots of the functions a) $y = x^2 - 3$, b) $y = x^2 + 2$, and c) $y = x^2 - 0.5$. (Problem 10.13a)

10.24 Reformulation of problem: Plot the polynomials for in the intervals $x^* - 2, x^* + 2]$, where x^* are the symmetry or anti-symmetry point of the polynomial. The point x^* is symmetry point if for any x, $p(x^* + x) = p(x^* - x)$, correspondingly x^* is antisymmetry point if $p(x^* + x) = -p(x^* - x)$. See the plots in figure below.

10.25 See plots below of piecewise polynomials.

Chapter 11

11.1 (a) $\{x \in R : x \neq \frac{1}{3} \text{ and } x \neq 1\}$

(b) $\{x \in R : x \neq 0 \text{ and } x \neq 2 \text{ and } x \neq -\frac{1}{2}\}$

(c) $\{x \in R : x \neq 0\}$

(d) $\{x \in R : x \neq 0 \text{ and } x \neq -\frac{3}{2}\}$

(e) $\{x \in R : x \neq \frac{2}{3} \text{ and } x \neq -4\}$

(f) $\{x \in R : x \neq -2 \text{ and } x \neq -1\}$

11.2 (a) $\sum_{i=1}^{101}(i + 1)x^i(x - 1)^i$ (Note misprint: 100 should be 102!)

(b) $\sum_{i=1}^{13} \frac{2^i}{x-i}$

11.3 (a) Note misprint: $f(x) = ax + b$ should be $f(x) = ax$. Proof: $f(x+y) = a(x + y) = ax + ab = f(x) + f(y)$.

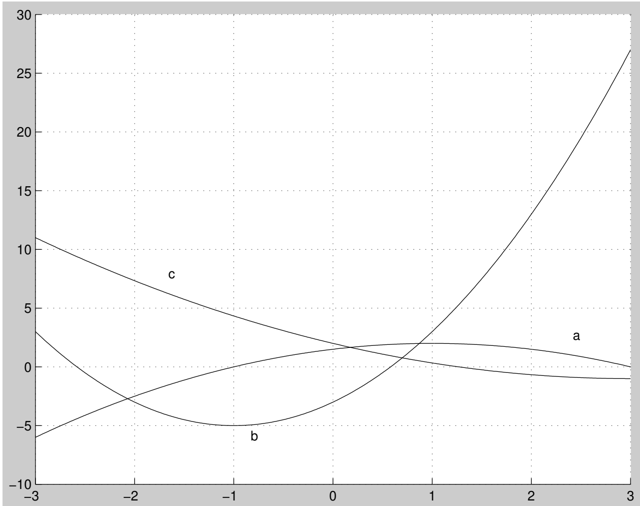


Figure 7: Plots of the functions a) $-\frac{1}{2}(x-1)^2+2$, b) $2(x+2)^2-5$ and c) $\frac{1}{3}(x-3)^2-1$ for $-3 \leq x \leq 3$. Note the x- and y-coordinates of the extreme points, (the points where the function has max or min value). (Problem 10.14)

(b) Proof: $g(x+y)=(x+y)^2=x^2+2xy+y^2$ and $g(x)+g(y)=x^2+y^2$, that is $g(x+y) \neq g(x)+g(y)$ unless $x=0$ or $y=0$.

11.4 (a) $\frac{x^2+2x-3}{x-1}=x+3$

(b) $\frac{2x^2-7x-4}{2x+1}=x-4$

(c) $\frac{4x^2+2x-1}{x+6}=4x-22+\frac{131}{x+6}$

(d) $\frac{x^3+3x^2+3x+2}{x+2}=x^2+x+1$

(e) $\frac{5x^3+6x^2-4}{2x^2+4x+1}=\frac{5}{2}x-2+\frac{\frac{11}{2}x-2}{2x^2+4x+1}$

(f) $\frac{x^4-4x^2-5x-4}{x^2+x+1}=x^2-x-4$

(g) $\frac{x^8-1}{x^3-1}=x^5+x^2+\frac{x^2-1}{x^3-1}$

(h) $\frac{x^n-1}{x-1}=x^{n-1}+x^{n-2}+\cdots+x^2+x=\sum_{i=1}^{n-1}x^i$

11.5 (a) $3(2x^2+1)-5=6x^2-2$

(b) $2(\frac{4}{x})^2+1=\frac{32}{x^2}+1$

(c) $\frac{4}{3x-5}$

(d) $3(2(\frac{4}{x})^2+1)-5=3(\frac{32}{x^2}+1)-5=\frac{96}{x^2}-2$

11.6 Note misprint: x/x^2 should be $1/x$. $f_1 \circ f_2 = 4(\frac{1}{x})+2=\frac{4}{x}+2$ and $f_2 \circ f_1=\frac{1}{4x+2}$ are *not* equal, for example for $x=1$.

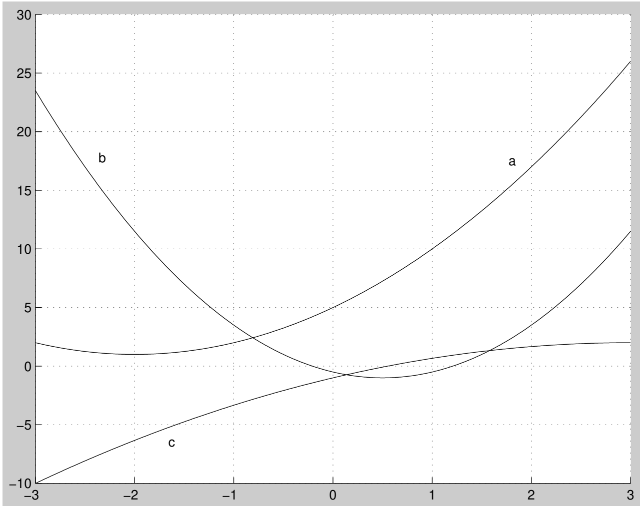


Figure 8: Plots of the functions a) $(x+2)^2+1$, b) $2(x-\frac{1}{2})^2-1$ and c) $-\frac{1}{3}(x-3)^2+2$ for $-3 \leq x \leq 3$. (Problem 10.15)

11.7 $f_1 \circ f_2 = a(cx+d)+b = acx+ad+b$ and $f_2 \circ f_1 = c(ax+b)+d = cax+cb+d$ are equal if and only if $ad+b = cb+d$, which is the case for example if $a=1$ and $c=1$, for any b and d, or otherwise if $d = \frac{cb-b}{a-1}$ or $b = \frac{ad-d}{c-1}$.

11.8 (a) $\{x \in R : x \neq 0 \text{ and } x \neq \frac{1}{4}\}$

(b) $\{x \in R : x \neq 1 \text{ and } x \neq x = \frac{1}{2} \text{ and } x = \frac{3}{2}\}$

Chapter 12

12.1 Since $|f(x_1) - f(x_2)| = |x_1^2 - x_2^2| = |x_1 + x_2||x_1 - x_2|$, we have $|f(x_1) - f(x_2)| \leq 16|x_1 - x_2|$ for $x_1, x_2 \in [-8, 8]$, and $|f(x_1) - f(x_2)| \leq 800|x_1 - x_2|$ for $x_1, x_2 \in [-400, 200]$.

12.2 For $a, b \in [10, 13]$ one has $|f(a) - f(b)| = |a^2 - b^2| = |(a+b)(a-b)| = |a+b||a-b| \leq 26|a-b|$

12.3 For $a, b \in [-2, 2]$ one has $|f(a) - f(b)| = |4a - 2a^2 - (4b - 2b^2)| = |4(a-b) - 2(a+b)(a-b)| = |(4 - 2a - 2b)(a-b)| = |4 - 2a - 2b||a-b| \leq 12|a-b|$, because $|4 - 2a - 2b| \leq 4 + 2|a| + 2|b| \leq 4 + 4 + 4 = 12$, for $a, b \in [-2, 2]$.

12.4 Since $|f(x_1) - f(x_2)| = |x_1^3 - x_2^3| = |x_1 - x_2||x_1^2 + x_1x_2 + x_2^2| \leq (4+4+4)|x_1 - x_2|$, we have $L = 12$.

12.5 Show that for all x_1, x_2, we have $||x_1| - |x_2|| \leq |x_1 - x_2|$. Thus $|f(x_1) - f(x_2)| = ||x_1| - |x_2|| \leq |x_1 - x_2|$ and $L = 1$.

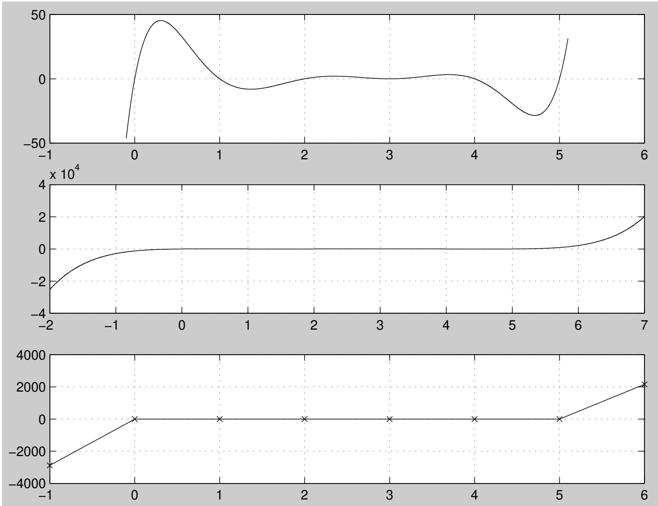


Figure 9: Three plots of the polynomial $360x - 942x^2 + 949x^3 - 480x^4 + 130x^5 - 18x^6 + x^7$, the top figure shows with matlab notation $x = -0.1 : 0.001 : 5.1$, the middle $x = -2 : 0.001 : 7$, and bottom $x = -1 : 6$. The matlab notation `x=x0:dx:x1`, means that x are the values starting with `x0` and increasing with interval `dx` until `x1` is reached. (Problem 10.21)

12.6 Realize (by plotting $f(x)$) that, given $|x_1 - x_2|$, $|f(x_1) - f(x_2)|$ attains its greatest value near $x_1 \approx x_2 \approx \pm 2$. Take $x_1 = 2$ and $x_2 = 2 - \epsilon$, where ϵ is a small number. Then show that $|f(x_1) - f(x_2)| \approx 32|x_1 - x_2|$.

12.7 For $a, b \in [1, 2]$ one has $|f(a) - f(b)| = |\frac{1}{a^2} - \frac{1}{b^2}| = |\frac{b^2-a^2}{a^2b^2}| = |\frac{(b+a)(b-a)}{a^2b^2}| = \frac{|a+b|}{a^2b^2}|a - b| \le 4|a - b|$, because $|a + b| \le 4$ and $a^2b^2 \ge 1$.

12.8 Show that $|f(x_1) - f(x_2)| \le \frac{|x_1+x_2|}{(1+x_1^2)(1+x_2^2)}|x_1 - x_2|$. For $x_1, x_2 \in [-2, 2]$ then show the Lipschitz continuity with $L = 4$. It is, however, possible to do better and get $L = 3\sqrt{3}/8$, which is the maximum value attained by $\frac{|x_1+x_2|}{(1+x_1^2)(1+x_2^2)}$, at $x_1 = x_2 = \pm 1/\sqrt{3}$. See the plot of this function below.

12.9 (a) $L = 100$

(b) $L = 10000$

(c) $L = 1000000$

12.10 For $x \neq y$ the Lipschitz inequality may be written $|f(x) - f(y)|/|x - y| \le L$. Let $x = 1/n, y = 1/2n, n = 1, 2, 3, \ldots$ and observe that $|f(x) - f(y)|/|x - y| = 2 * n^2$, which is greater than any L for $n > \sqrt{L/2}$.

12.11 (a) For $x \neq y$ one can write the Lipschitz inequality as $|f(x) - f(y)|/|x -$

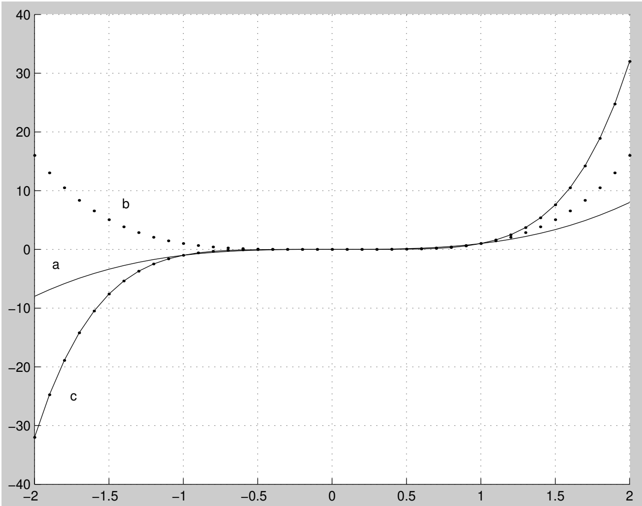


Figure 10: Plots of a) x^3, b) x^4 and c) x^5. Note that x^3 and x^5 are odd while x^4 is even. (Problem 10.23)

$y| \leq L$. With $x = 0$ and $y = -1/N$ we have $f(x) - f(y)|/|x - y| = |0 - 1|/|0 - (-1/N)| = N$, which is larger than any L for $N > L$.

(b) Yes!

12.12 If the Lipschitz constant L is extremly large then the function is close to discontinuous from a practical point of view.

12.13 Note misprint: $f_2 - f_2$ should be $f_1 - f_2$ and the Lipschitz constant of cf_1 should be $|c|f_1$. We have $|(f_1(x) - f_2(x)) - (f_1(y) - f_2(y))| \leq |f_1(x) - f_1(y)| + |f_2(x) - f_2(y)| \leq (L_1 + L_2)|x - y|$ and $|cf_1(x) - cf_1(y)| \leq |c|\,|f_1(x) - f_1(y)| \leq |c|L|x - y|$.

12.14 Note first that a Lipschitz constant for $f(x) = x^n$ on $[-c, c]$ is nc^{n-1}, see Problem 12.14. Then using Theorem 12.1 we readily obtain the desired result.

12.15 From 12.10 it follows that $1/x$ is not Lipschitz on $(0, 1]$ and thus it can not be Lipschitz on $[-11]$ since $(0, 1] \subset [-1, 1]$.

12.16 Observe that $1/f_2$ is Lipschitz mith Lipschitz constant $1/m^2$, since $|1/f_2(x) - 1/f_2(y)| \leq |f_2(x) - f_2(y)|/m^2$. Now the Theorem follows from Theorem 12.4.

12.17 (a) Lipschitz with $L = 138$ using the formula in 12.14.

(b) Lipschitz with $L = 16/9$.

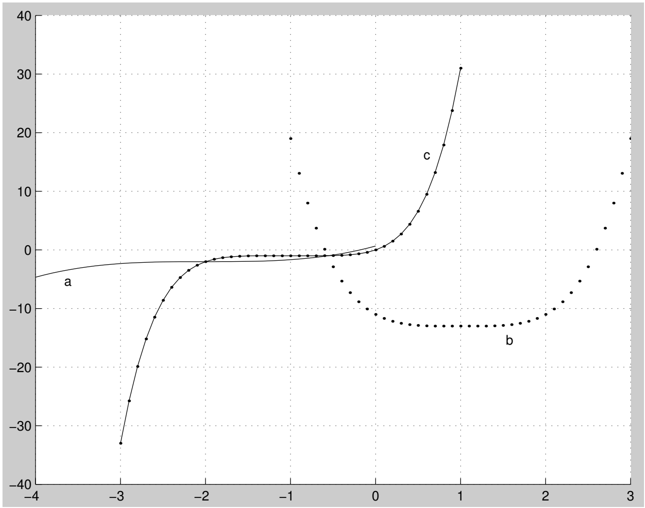


Figure 11: a) is plotted in interval [-4,0], b) is plotted in [-1,3], and c) in [-3,1]. (Problem 10.23)

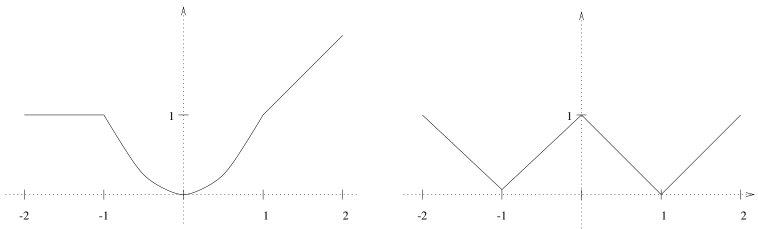


Figure 12: To the left problem a) and to the right b). (Problem 10.25)

(c) Not Lipschitz continuous by Theorem 12.3, because not bounded on the given interval.

(d) Lipschitz with $L = 32$, use Theorem 12.6.

12.18 Follows from Theorem 12.5 because $c_1 x + c_2(1-x) \geq \min(c_1, c_2) > 0$ for $x \in [0,1]$.

Chapter 13

13.1

13.2

13.3 (a) $\{3^i\}_{i=0}^{\infty}$

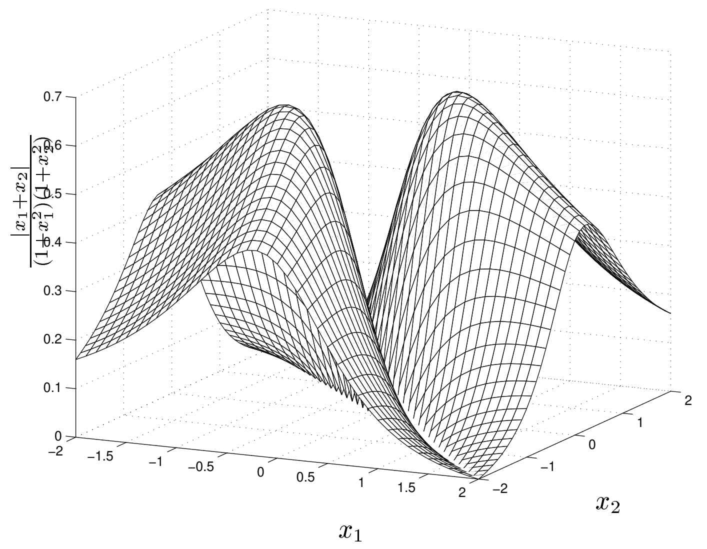


Figure 13: A plot of the function $\frac{|x_1+x_2|}{(1+x_1^2)(1+x_2^2)}$ in $[-2,2]\times[-2,2]$. (Problem 12.8)

(b) $\{4^i\}_{i=2}^{\infty}$

(c) $\{(-1)^i\}_{i=2}^{\infty}$

(d) $\{1+3i\}_{i=1}^{\infty}$

(e) $\{3i-1\}_{i=1}^{\infty}$

(f) $\{5^{-i}\}_{i=-3}^{\infty}$

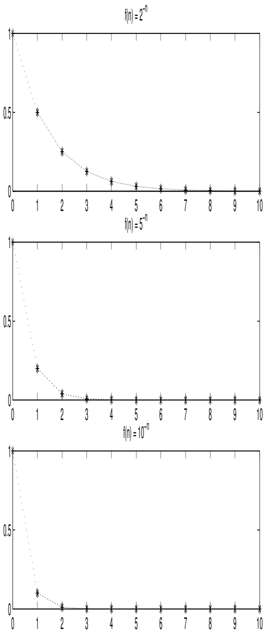
13.4 (a) $|\frac{8}{3n+1}-0| = \frac{8}{3n+1} \le \epsilon$ if $3n+1 \ge \frac{8}{\epsilon}$,
that is if $n \ge N$ where $N = \frac{8/\epsilon - 1}{3}$.

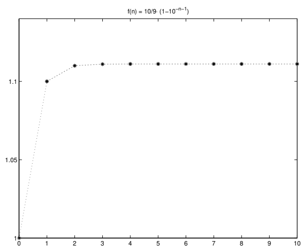
13.5 $|r^n - 0| \le \epsilon$ if $(\frac{1}{2})^n \le \epsilon$, that is if $2^n \ge \frac{1}{\epsilon}$.

13.6

13.7 (a) Choose any $M > 0$. Now we have to show that there exists an N such
that $-4n+1 < -M$ for all $n > N$. We see that this is true for $N = (M+1)/4$.

(b) If $\lim_{n\to\infty} n^3 = \infty$ then surely
$\lim_{n\to\infty} n^3 + n^2 = \infty$, since $n^3 < n^3 + n^2$
for $n \ge 1$. So it is sufficient to, for any $M > 0$, find an N such that $n^3 > M$, for $n > N$. This is true for $N = M^{1/3}$.


f(n) = 2^{-n}
f(n) = 5^{-n}
f(n) = 10^{-n}


f(n) = 10/9· (1−10^{−n−1})

13.8 *Correction*: Should be $r > 1$, not $|r| \geq 2$.

For any $M > 0$, we want to show that there exists an N such that $r^n > M$ for all

$n > N$. But $r^n > M \Leftrightarrow n \ln r > \ln M \Leftrightarrow n > \ln(\frac{M}{r})$,

so let $N = ln(\frac{M}{r})$.

13.9 (a) $\frac{1}{1-(-.5)} = 2/3$

(b) $3\frac{1}{1-\frac{1}{4}} = 4$

(c) $\frac{5^(-2)}{1-\frac{1}{5}} = \frac{1}{20}$.

13.10 (a) $\frac{1}{1-r^2}$

(b) $\frac{1}{1-(-r)} = \frac{1}{1+r}$.

13.11 All are equal to $\{\frac{2}{3}, \frac{4}{5}, \frac{8}{3}, \frac{16}{5}, \dots\}$

except (e) which is $\{\frac{4}{5}, \frac{8}{3}, \frac{16}{5}, \dots\}$.

13.12 (a) $\left\{\frac{2+(n+5)^2}{9^{n+5}}\right\}_{n=-4}^{\infty}$

(b) $\left\{\frac{2+(n-2)^2}{9^{n-2}}\right\}_{n=3}^{\infty}$

(c) $\left\{\frac{2+(n-1)^2}{9^{n-1}}\right\}_{n=2}^{\infty}$

13.13 STRYKES, triangelolikheten bör behandlas redan i kap

6, rational numbers.

The task is to prove the *triangle inequality*: $|a + b| \leq$

—a—+—b—$forall$a,b.

If $a \geq 0$, $b \geq 0$, then $|a + b| = a + b = |a| + |b|$.

If $a \leq 0$, $b \geq 0$, $a + b \geq 0$, then $|a + b| = a + b \leq -a + b = |a| + |b|$.

If $a \leq 0$, $b \geq 0$, $a + b \leq 0$, then $|a + b| = -a - b \leq -a + b = |a| + |b|$.

The remaining cases are proved in a similar way.

Proof of (13.7): $|a - b| = |a - c + c + b| \leq |a - c| + |c + b|$.

13.14 The triangle inequality gives $|(a_n - b_n) - (A - B)| = |(a_n - A) - (b_n - B)| \leq$

—a$_n - A| + |b_n - B|$, where the right side can be made as small as desired

by taking n sufficiently large. Another proof can be found in

Section 13.5.

13.15 STRYKES ??

13.16 If n is sufficiently large, then

$|a_n - A| \le \frac{1}{2}|A|$ and $|b_n - B| \le \frac{1}{2}|B|$, so that

$|a_n| = |a_n - A + A| \le |a_n - A| + |A| \le \frac{3}{2}|A|$, and

$|B| = |B - b_n + b_n| \le |B - b_n| + |b_n| \le \frac{1}{2}|B| + |b_n|$, and

$|b_n| \ge \frac{1}{2}|B|$, and $\frac{1}{|b_n|} \le \frac{2}{|B|}$. For large

n we thus get

$\Big|$

a$_n \frac{}{b_n - \frac{A}{B}} \Big| = \Big| \frac{a_n}{b_n} -$

a$_n \frac{}{B + \frac{a_n}{B} - \frac{A}{B}} \Big| = \Big|$

a$_n \frac{}{b_n B(B - b_n) + \frac{1}{B}(a_n - A)} \Big|$

$\le \frac{|a_n|}{|b_n|\,|B|}|B - b_n| + \frac{1}{|B|}|a_n - A| \le$

3 —A— $\frac{}{|B|^2|b_n - B| + \frac{1}{|B|}|a_n - A|,}$

where the right side can be made as small as desired
by taking n sufficiently large.

Another proof can be found in Section 13.5.

13.17 (a) 1

(b) divergent to $+\infty$, because $a_n = n^2(4 - 6n^{-1}) \ge n^2$ for $n \ge 2$

(c) 0, because $|a_n - 0| = n^{-2}$

(d) 1/3

(e) divergent, because $a_n = \frac{(-1)^n}{7 + n^{-2}}$ flips (approximately) between $\frac{1}{7}$ and $-\frac{1}{7}$ when n is large

(f) 2

(g) -4 (all a_n equal -4)

(h) $-5/8$

(i) divergent to $+\infty$, because $a_n = n\frac{2 + n^{-2} + n^{-3}}{6 - 5n^{-2}} \ge 2n$

(j) -1

13.18 (a) 2^{-37}

(b) 7^4

(c) 2^{-8}

(d) 1

13.19 $f(a_n)$ with $f(x)$ and a_n given by

(a) $f(x) = \left(\frac{x^2+2}{x^2+1}\right)^3$, $a_n = n$;
$f(x) = \left(\frac{x+2}{x+1}\right)^3$, $a_n = n^2$;
$f(x) = x^3$, $a_n = \frac{n^2+2}{n^2+1}$;

(b) $f(x) = x^4 + x^2 + 1$, $a_n = n^2$;
$f(x) = x^8 + x^4 + 1$, $a_n = n$;
$f(x) = x + 1$, $a_n = n^8 + n^4$;

13.20 $|0.99\cdots 99_n - 1| = (0.1)^n = 0.00\cdots 01_n \leq \epsilon$
for $n \geq N$, if N is
the index of the first non-zero decimal in ϵ.
Note also that, using the geometric sum,
$0.99\cdots 99_n = 0.9\sum_{k=0}^{n-1}$
$(0.1)^k = 0.9\frac{1-(0.1)^n}{1-0.1} = 1 - (0.1)^n$.

13.23 STRYKES.
This problem has been moved to Problem 13.2.

13.24 FLYTTAS. This problem should be moved to Chapter 13.
$\sqrt{2^2 - 1} = \sqrt{3}$

Chapter 14

14.2 (a) See (b).

(b) Assume $\sqrt{p} = a/b$ where largest common divisor of a and b is 1. Then $b^2 p = a^2$, and since p is prime $a = p\alpha$ for some α, and thus $b^2 p = p^2\alpha^2$ or $b^2 = p\alpha^2$ and thus $b = p\beta$. This is a contradiction, since p divides both a and b. Make sure you understand all details.

14.3 $4^{1/3}, 3^1/4, 4^1/4$, etc.

14.8 Assumption give $|b| = b < b - a < c$ and $|a| = -a = (b-a) - b < c - b < c$

Chapter 15

15.2 (b) Have that $|xy - x_i y_i| = |(x - x_i)y + x_i(y - y_i)| \le |(x - x_i)y| + |x_i(y - y_i)| = |y||x - x_i| + |x_i||y - y_i| \le |y|2^{-i} + (|x| + 0.1)2^{-i}$, where we used the fact $|x_i| = |(x_i - x) + x| \le |x_i - x| + |x| \le 2^{-i} + |x| \le 0.1 + |x|$ for $i \ge 4$.

15.3 $x = 0.373737...$ and $y = \sqrt{2} = 1.414213..$ give $x_1 y_1 = 0.3 \times 1.4 = 0.42$, $x_2 y_2 = 0.37 \times 1.41 = 0.5217$, $x_3 y_3 = 0.373 \times 1.414 = 0.527422$, etc.

15.4 No, because if the limit x would be less than 1 then $d = (1 - x)/2$ is positive, and $\frac{i}{i+1} = \frac{i+1-1}{i+1} = 1 - \frac{1}{i+1} \ge 1 - d = x + d$ for $\frac{1}{i+1} \le d$, that is for $i \ge \frac{1}{d} - 1$, which contradict the assumption that $\{\frac{i}{i+1}\}$ converges to x.

15.7 (a) $\{x \in R : -2\sqrt{2} \le x \le 2\sqrt{2}\}$

(b) $\{x \in R : x < 2\sqrt{2} - 2/3 \text{ or } x > 2\sqrt{2} + 2/3\}$

15.9 (a) The sequence is $\{\frac{1}{i^2}\}$ (ok to shift the index since we are only concerned with the *limit*). For $|\frac{1}{i^2} - \frac{1}{j^2}| = |\frac{j^2 - i^2}{i^2 j^2}| = \frac{|j^2 - i^2|}{i^2 j^2} \le \frac{j^2 + i^2}{i^2 j^2} = \frac{1}{i^2} + \frac{1}{j^2} \le \epsilon$ if $i, j \ge N$ and $N = \frac{1}{\sqrt{2\epsilon}}$.

15.10 Assume that i^2 is a Cauchy sequence. Choose $\epsilon > 0$ and N, and take $j = N$ and $i = j + 1$. Compute $|i^2 - j^2|$ and derive a contradiction.

15.11 (b) $1/3$

15.12 (b) *Hint:* You need to show that a Cauchy sequence is bounded. See also 15.2(b)

15.14 Let $\bar{c}$ denote the smallest of all c:s. Choose an $\epsilon > 0$. Then $\bar{c} - \epsilon \le x_i \le \bar{c}$ for all $i > N(\epsilon)$. So $\bar{c}$ is the limit by the formal definition.

15.15 $\sqrt{2} = 1.414...$ gives $f(1.4) = \frac{1.4}{1.4+2} = 0.4117647....$, $f(1.41) = \frac{1.41}{1.41+2} = 1.41348973...$, $f(1.414) = 0.4141769185...$, etc. (Hmm, looks familiar, like $\sqrt{2} - 1$. Could it be that $f(\sqrt{2}) = \frac{\sqrt{2}}{\sqrt{2}+2} = \sqrt{2} - 1$? Check!).

15.16 6

15.12 $L = \frac{n}{0.01^{2n}}$

15.20 (a) $(-2, 4]$

(b) $(-3, -1) \cup (-1, 2]$

(c) $[-2, -2] \cup [0, \infty)$

(d) $(-\infty, 0) \cup (1, \infty)$

15.22 $[2, 3)$

15.26 $S = 0.0106020716$

15.28 $R = \{1, 9\}$

15.33 For $a, b > \delta$ one has $|f(a) - f(b)| = |\sqrt{a} - \sqrt{b}| = |\frac{(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})}{\sqrt{a}+\sqrt{b}}| = |\frac{a-b}{\sqrt{a}+\sqrt{b}}| = \frac{|a-b|}{\sqrt{a}+\sqrt{b}} \le L|a-b|$ where $L = \frac{1}{2\sqrt{\delta}}$

15.36 (a) $60 = 2 \times 2 \times 3 \times 5$

(b) $96 = 2 \times 2 \times 2 \times 2 \times 2 \times 3 = 2^5 \times 3$

(c) $112 = 2^4 \times 7$

(d) $129 = 3 \times 43$

Chapter 16

Problems

16.1 Use the Bisection Algorithm to find a solution, accurate to within 10^{-2}, to the equation $x + 0.5 + 2\cos \pi x = 0$ on the interval $[0.5, 1.5]$.

16.2 Use the Bisection Algorithm to find an approximation to $\sqrt{3}$ that is accurate to within 10^{-4}.

16.3 Find a bound for the number of iterations needed to approximate a solution to the equation $x^3 + x - 4 = 0$ on the interval $[1, 4]$ to an accuracy of 10^{-3}.

16.4 A trough of water of length $L = 10$ feet has a cross section in the shape of a semicircle with radius $r = 1$ foot. When filled with water to within a distance h of the top, the volume $V = 12.4 ft^3$ of the water is given by the formula

$$12.4 = 10[0.5\pi - \arcsin h - h(1 - h^2)^{1/2}].$$

Determine the depth of the water to within 0.01 feet.

16.5 Suppose $f(x)$ has only simple roots in (a, b). If $f(a)f(b) < 0$, show that there are an odd number of roots of $f(x) = 0$ in (a, b). If $f(a)f(b) > 0$, show that there are an even number (possibly zero) of roots of $f(x) = 0$ in (a, b).

16.6 Show that the Bisection method converges linearly, that is, $\lim_{n\to\infty} \frac{r_{n+1}-r}{r_n - r}$ is constant.

16.7 Find all the roots of the function $f(x) = \cos x - \cos 3x$.

16.8 Find the root or roots of $\ln[(1 + x)/(1 - x^2)] = 0$.

16.9 Find where the graphs of $y = 3x$ and $y = e^x$ intersect by finding roots of $e^x - 3x = 0$ correct to four decimal digits.

16.10 Consider the bisection method, determine how many steps are required to guarantee an approximation of a root to six decimal places (rounded).

16.11 By graphical methods, locate approximations to all roots of the nonlinear equation $\ln(1+x)+\tan(2x)=0$.

16.12 Equation $xe^x - 2 = 0$ has a simple root r in $[0,1]$. Use the bisection method to estimate r within seven decimal digits.

16.13 Use the bisection method to find, as accurately as you can, all real roots for each equation.

(a) $x^3 - x^2 - x - 1 = 0$

(b) $x^2 = e^{-x^2}$

(c) $\ln|x| = \sin x$

16.14 A certain technical problem requires solution of the equation

$$21.13 - \frac{3480}{T} - 5.08 \log T = 0$$

for a temprature T. Technical information indicates that the temperature should lie between $400°$ and $500°$. Use the bisection method to estimate the desired temperature to nearest degree.

16.15 Use the bisection method with some calculus to find the minimum value of $f(x) = \sin x/x$ on interval $[\pi, 2\pi]$.

Answers

16.1 $r_7 = 0.711$

16.2 $\sqrt{3} \approx r_{14} = 1.7320$

16.3 $r_{12} = 1.3787$

16.4 $h \approx r_{13} = 0.1617$ so the dept is $r - h \approx 1 - 0.1617 = 0.838$ feet

16.7 $\{0, \pm\pi/2, \pm\pi, \pm 3\pi/2, \pm 2\pi. \ldots\}$

16.8 $x = 0$

16.9 0.61906, 1.51213

16.10 20 steps

16.11 $\{0,\ \frac{\pi}{4}+\varepsilon,\ \frac{3\pi}{4}+\varepsilon,\ \frac{5\pi}{4}+\varepsilon, \ldots\}$, where ε starts at approximatelt 1/2 and decreases.

16.12 $r = 0.8526055$

16.14 $475°$

Chapter 18

18.2 Yes $L = 1,\ \theta = 1/3$

18.3 $(b),\ (c)$

18.4 No

18.6 No

Chapter 19

19.1 Break even if sales x equals expenses $100 + 0.2x$, that is if $x = g(x)$ where $g(x) = 100 + 0.2x$.

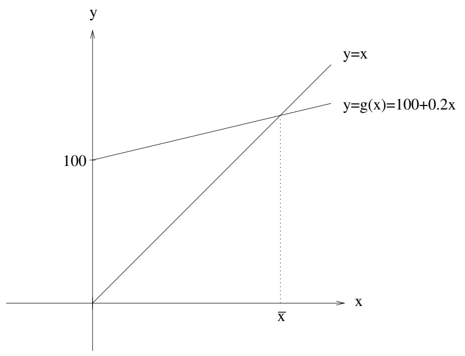


Figure 14: Problem 19.1

19.2 (a) For example $f(x) = \frac{x^3-1}{x+2} - x = 0$ or $f(x) = x^3 - 1 - x(x+2) = x^3 - x^2 - 2x - 1 = 0$

(b) For example $f(x) = x^5 - x^3 + 4 - x = 0$ or $f(x) = \frac{x^5-x^3}{x-4} - 1 = 0$.

19.3 (a) For example $x = g(x) = x - 0.1(7x^5 - 4x^3 + 2)$ or $x = g(x) = (\frac{7}{4}x^5 + \frac{1}{2})^{1/3}$

(b) For example $x = g(x) = x + 0.3(\frac{2}{x} - x^3)$ or $x = g(x) = \frac{2}{x^3}$ or $x = g(x) = (3x + \frac{2}{x^3})/4$.

19.4 Skippas

19.5 (a) $[0, 3]$ (For $f(x) = \frac{x^3-1}{x+2}$ one has $f(0) = -\frac{1}{2} < 0$)

(b) $[-2, 0]$ (For $f(x) = x^5 - x^3 + 4 - x$ one has $f(-2) = -18 < 0$, $f(0) = 4 > 0$).

19.6

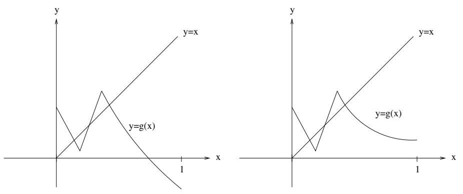


Figure 15: Problem 19.6

19.7
```
function x = fixedpoint(g, x0, max_iter, tol)
iter = 0;
xold = x0;
x = x_old;
x_new = eval(g);
while iter < max_iter & abs(x_new - x_old) > tol
  x = x_old;
  x_new = eval(g);
  iter = iter + 1;
end
x = x_new;
```

19.8 Rewrite the equation $x(0.02+2x)^2 = 1.8\,10^{-5}$ as $100\,x(2+2\,100\,x)^2 = 18$, and rescale by introducing $y = 100x$ to obtain $y(2+2y)^2 = 18$. Writing this as $y = g(y) = \frac{18}{(2+2y)^2}$, the fixed point iterations $y_{j+1} = g(y_j)$ does not converge. Try instead $y = g(y) = (y + \frac{18}{(2+2y)^2})/2$ for which the iteration $y_{j+1} = g(y_j)$ with $y_0 = 2$ gives the sequence
2.00000000000000
1.06944444444444
1.06010396178350
1.06020866771593
1.06020713377205
1.06020715618506
1.06020715585756
1.06020715586235
that is, $x = y/100 \approx 0.010602071559$.

19.9 Rewrite the equation $x(0.037+2x)^2 = 1.57\,10^{-9}$ with $y = 1000x$ as $y(37+2y)^2 = 1.57$. Write this as $y = g(y) = \frac{1.57}{(37+2y)^2}$ and compute $y_{j+1} = g(y_j)$ with $y_0 = 1$ to obtain
1.00000000000000
0.00114669453395
0.00114668034327
0.00114668034503
that is, $x = y/1000 \approx 0.0000011466803$.

19.10 Rewrite equation $1 = \frac{4^2 R}{(3+R)^2}$ with $x = R$ as $x = g(x) = x - 1.5(\frac{16x}{(3+x)^2} - 1)$ and compute iterates $x_{j+1} = g(x_j)$ with $x_1 = 2$.

This gives the sequence of iterates

2.00000000000000
1.27225415228543
1.09935039113477
1.02928661189918
1.00771903960222
1.00195760369188
1.00049119586678
1.00012291204137
1.00003073509157
1.00000768421569
1.00000192108160
1.00000048027213
1.00000012006814
1.00000003001704
1.00000000750426
1.00000000187607
1.00000000046902
1.00000000011725
1.00000000002931

converging to 1.

Iterating with $g(x) = x + 20(\frac{16x}{(3+x)^2} - 1)$ and $x_0 = 8$ gives a sequence of iterates

8.00000000000000
8.98369412815009
9.00182407284279
8.99979747927866
9.00002250420120
8.99999749955665
9.00000027782733
8.99999996913030
9.00000000342996
8.99999999961889
9.00000000004234

converging to 9. (Equation can also be solved analytically for R)

19.11 Rewrite the equation $(2 + \frac{50}{V^2})(V - 0.011) = 3 \times 15$ with $x = V$ as $x = g(x) = 0.011 + 45/(2 + 50/x^2)$ and compute iterates $x_{j+1} = g(x_j)$ with $x_0 = 20$ to obtain

20.00000000000000
21.32406304182749
21.33843041035536
21.33992672744253
21.34008240256070
21.34009859707175
21.34010028172915
21.34010045697786
21.34010047520834
21.34010047710479
21.34010047730207
21.34010047732259

19.12 Proof by induction: True for $n = 3$. Assume true for n. Then

$$x_{n+1} = \frac{1}{4}x_n + \frac{1}{4} = \frac{1}{4}(\frac{1}{4^n}x_0 + \sum_{i=1}^{n} \frac{1}{4^i}) + \frac{1}{4}$$
$$= \frac{1}{4^{n+1}x_0} + \sum_{i=2}^{n+1} \frac{1}{4^i} + \frac{1}{4} = \frac{1}{4^{n+1}x_0} + \sum_{i=1}^{n+1} \frac{1}{4^i},$$

showing that formula valid for $n+1$.

19.13 (a) $x_n = 2^n x_0 + \frac{1}{4}\sum_{i=1}^{n} 2^i$

(b) For any given $M > 0$ we have that $x_n > M$ if n is large enough, because $\sum_{i=0}^{n} 2^i = (1-2^{n+1})(1-2) = 2^{n+1} - 1$.

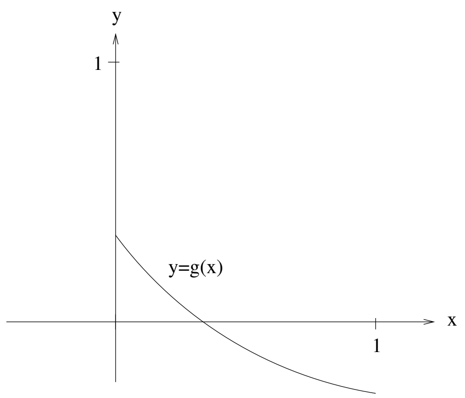
19.16 

Figure 16: Problem 19.16.

19.14 (a) $x_n = (\frac{3}{4})^n + \sum_{i=1}^{n} \frac{3^{i-1}}{4^i}$

(b) First we point out that $lim_{n\to}\inf(\frac{3}{4})^n \to 0$. Then we look at the sum $\sum_{i=1}^{n} \frac{3^{i-1}}{4^i} = \frac{1}{3}\sum_{i=1}^{n} (\frac{3}{4})^i = \frac{1}{3}\frac{1-(\frac{3}{4})^(n+1)}{1-\frac{3}{4}} = \frac{1}{3}\frac{1}{1-\frac{3}{4}} = \frac{4}{3}$

19.15 $x_n = m^n x_0 + b\frac{1-m^n}{1-m}$

We show that $g(x) = mx+b$ is a contraction mapping: $|g(x)-g(y)| = |mx+b-(my+b)| = |m||x-y| = L|x-y|$. $L < 1$ so g is a contraction mapping and the fixpoint iteration has a unique solution by Theorem 19.1. The solution is $\bar{x} = b/(1-m)$.

19.16 Draw for example the function $g(x) = 2x - 3$ for which $g(x) \in [0, 1]$, when $x \in [1.5, 2]$.

19.17 Need to know the specific fixpoint functions used in the 19.3 problem to solve this problem.

19.18 If $g'(x)$ is bounded in the interval then $g(x)$ is Lipschitz continious in the interval. $g'(x) = \frac{2x}{(1+x^2)^2} \Rightarrow L = \max_{x\in[a,b]} |\frac{2x}{(1+x^2)^2}| \leq 1 \Rightarrow g : [a, b] \to [a, b]$, that is g is a contraction mapping. By theorem 19.2 we now have that if the starting point in the fixed point iteration $x_0 \in [a, b]$ then the sequence given by the iteration converges to a $\bar{x} \in [a, b]$.

19.19 Using $|x_{k+1} - x_k| \leq L^k|x_1 - x_0|$ we compute $(|x_{k+1} - x_k|/|x_1 - x_0|)^{1/k}$ for $k = 1, 2, 3, 4$ with data from the table. The result is 0.875 for $k = 1, 2, 3, 4$, hence $L = 0.875$.

19.20 If $g'(x)$ is bounded in the interval then $g(x)$ is Lipschitz continious in the interval. $g'(x) = \frac{4x^3(10-x)^2+2(10-x)x^4}{(10-x)^4} \Rightarrow$ $L = \max_{x\in[-1,1]} |\frac{4x^3(10-x)^2+2(10-x)x^4}{(10-x)^4}| = |\frac{4}{(10-1)^2} + \frac{2}{(10-1)^3}| = \frac{38}{729} < 0.053 < 1 \Rightarrow g$ is a contraction mapping.By theorem 19.2 we now have that if the starting point in the fixed point iteration $x_0 \in [-1, 1]$ then the sequence given by the iteration converges to a $\bar{x} \in [-1, 1]$. g is not a contraction mapping in [-9.9,9.9]. $L = \max_{x\in[-9.9,9.9]} | \frac{4x^3(10-x)^2+2(10-x)x^4}{(10-x)^4}| = |\frac{4\cdot 9.9^3}{(10-9.9)^2} + \frac{2\cdot 9.9^4}{(10-9.9)^3}| < 2\cdot 10^7$, which is larger than 1.

19.21 Using the method from Problem 19.19 we get the estimates for the Lipschitz constant to be 0.6954, 0.6152, 0.5867, 0.5683 for $k = 1, 2, 3, 4$, respectively. Alternatively we can compute $|x_{k+1} - x_k|/|x_k - x_{k-1}|$ which gives 0.6954 , 0.5443 , 0.5334 , 0.5165 for $k = 1, 2, 3, 4$, respectively. Both these computations show that the convergence is not linear.

19.22 (a) If $g'(x)$ is bounded in the interval then $g(x)$ is Lipschitz continious in the interval. $g'(x) = 2x^2$. $L = \max_{x\in[-1/2,1/2]} |g'(x)| = |2 \cdot (\frac{1}{2})^2| = 0.5$.
(b)
(c) $x_i = g(x_{i-1}) = g(\bar{x} + x_{i-1} - \bar{x}) = \frac{2}{3}(\bar{x} + (x_{i-1} - \bar{x}))^3$. Using the fact that $\bar{x} = 0$ we get: $|x_i - \bar{x}| = |\frac{2}{3}(x_{i-1} - \bar{x})^3|$

19.23 Use amongst other things that $x_{i-1} \approx \sqrt{2}$.

19.24 (a) $x^2 + x - 6 = 0 \to x(x + 1) = 6 \to x = \frac{6}{(x+1)}$. The error is estimated by $|x_i - \bar{x}| = |g(x_{i-1}) - g(\bar{x})| = \frac{6}{x_{i-1}+1} - \frac{6}{\bar{x}+1} \leq |\frac{6(\bar{x}-x_{i-1})}{(\bar{x}+1)^2}| \leq \frac{2}{3}|\bar{x} - x_{i-1}|$, when the sequense of the fixed point iteration has converged and $x_{i-1} \leq \bar{x}$.
(b) $x^2 + x - 6 = 0$ adding x^2 on each side gives: $2x^2 + x = x^2 + 6 \to x = \frac{x^2+6}{2x+1}$. The error is estimated using $\bar{x} = 2$ and

$x_i = g(\bar{x} + (x_{i-1} - \bar{x})) = \frac{\bar{x}+(x_{i-1}-\bar{x})^2+6}{2(\bar{x}+(x_{i-1}-\bar{x}))+1} = \frac{4x_{i-1}+2+(x_{i-1}-\bar{x})^2}{4+1}$.
$|x_i - \bar{x}| = |2 + \frac{(x_{i-1}-\bar{x})^2}{5} - 2| = |\frac{(x_{i-1}-\bar{x})^2}{5}|$

19.25 The equation for the line through the points $(x_{i-1}, f(x_{i-1}))$ and $(x_i, f(x_i))$ is given by $y = f(x_{i-1}) + \frac{x-x_{i-1}}{x_i-x_{i-1}}(f(x_i) - f(x_{i-1})$ which for $y = 0$ has the solution $x_{i+1} = x_{i-1} - f(x_i)(x_i - x_{i-1})/(f(x_i) - f(x_{i-1}))$. Convergence factor?

Chapter 20

20.1 Proof (example, analytical) of $\lambda(\mu a) = (\lambda\mu)a$: By definition

$$\lambda(\mu a) = \lambda(\mu(a_1, a_2)) = \lambda(\mu a_1, \mu a_2) = (\lambda(\mu a_1), \lambda(\mu a_2)).$$

and

$$(\lambda\mu)a = (\lambda\mu)(a_1, a_2) = ((\lambda\mu)a_1, (\lambda\mu)a_2).$$

The desired identity thus follows from the associative law for real number multiplication.

20.2 $f(x) = x + 2(P_a(x)x - x) = 2P_a(x) - x = 2\frac{a\cdot x}{|a|^2}\,a - x$. The corresponding matrix is

$$\begin{pmatrix} 2\frac{a_1^2}{|a|^2} - 1 & 2\frac{a_1 a_2}{|a|^2} \\ 2\frac{a_1 a_2}{|a|^2} & 2\frac{a_2^2}{|a|^2} - 1 \end{pmatrix}$$

20.3 (a) $\sqrt{13}$ (b) $\sqrt{17}$ (c) $\sqrt{52}$ (d) $\sqrt{8}$ (e) $(3,2)/\sqrt{13}$ (f) $(1,4)/\sqrt{17}$

20.4 $|\frac{a}{|a|}| = |(\frac{a_1}{|a|}, \frac{a_2}{|a|})| = \sqrt{(\frac{a_1}{|a|})^2 + (\frac{a_2}{|a|})^2} = \sqrt{\frac{a_1^2}{|a|^2} + \frac{a_2^2}{|a|^2}} = \sqrt{\frac{a_1^2+a_2^2}{|a|^2}} = \sqrt{1} = 1$

20.5 (b) $a\cdot b = |a||b|\cos(\theta) \leq |a||b|$. (a) $|a+b|^2 = (a_1+b_1)^2+(a_2+b_2)^2 = a_1^2+a_2^2+b_1^2+b_2^2+2a_1b_1+2a_2b_2 = |a|^2+|b|^2+2a\cdot b \leq |a|^2+|b|^2+2|a||b| = (|a|+|b|)^2$

20.6 (a) 7 (b) 5 (c) 0

20.7 (a), (c) and (e) makes sense.

20.8 $\theta = \arccos(\frac{10}{\sqrt{2}\sqrt{58}})$

20.9 All $a = (a_1, a_2)$ such that $2a_1 + a_2 = 2$. A line in the a_1, a_2 plane, with normal $(2,1)$ passing through, for example, the point $(0,2)$.

20.10 (a) $P_b(a) = \frac{b\cdot a}{|b|^2}\,b = \frac{5}{5}(1,2) = (1,2)$ (b) $P_b(a) = \frac{0}{5}(1,2) = (0,0)$ (c) $P_b(a) = \frac{6}{5}(1,2)$ (d) $P_b(a) = \frac{3\sqrt{2}}{5}(1,2)$

20.11 $b = c + d$ where $d = b - c$ and (a) $c = P_a(b) = \frac{a\cdot b}{|a|^2}\,a = \frac{13}{5}(1,2)$ (b) $c = \frac{-1}{5}(-2,1)$ (c) $c = \frac{16}{8}(2,2)$ (d) $c = \frac{8\sqrt{2}}{4}(\sqrt{2},\sqrt{2})$

20.12 $|c|^2 = |a-b|^2 = (a_1-b_1)^2 + (a_2-b_2)^2 = a_1^2 + a_2^2 + b_1^2 + b_2^2 - 2(a_1b_1 + a_2b-2) = |a|^2 + |b|^2 - 2a \cdot b = |a|^2 + |b|^2 - 2|a||b|\cos(\phi)$

20.13 See previous problem.

20.14 (a) $Ax = (5,11)^\top$ and $A^\top x = (7,10)^\top$ (b) $Ax = (3,7)^\top$ and $A^\top x = (4,6)^\top$

20.15 (a) $\begin{pmatrix} 19 & 22 \\ 43 & 50 \end{pmatrix}$ (b) $\begin{pmatrix} 23 & 34 \\ 31 & 46 \end{pmatrix}$ (c) $\begin{pmatrix} 26 & 30 \\ 38 & 44 \end{pmatrix}$ (d) $\begin{pmatrix} 17 & 23 \\ 39 & 53 \end{pmatrix}$
(e) $\begin{pmatrix} 19 & 43 \\ 22 & 50 \end{pmatrix}$ (f) $\begin{pmatrix} 19 & 43 \\ 22 & 50 \end{pmatrix}$ (g) $\begin{pmatrix} -2 & 1 \\ 1.5 & -0.5 \end{pmatrix}$ (h) $\begin{pmatrix} -4 & 3 \\ 3.5 & -2.5 \end{pmatrix}$
(i) $\begin{pmatrix} 12.5 & -5.5 \\ -10.75 & 4.75 \end{pmatrix}$ (j) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

20.16 The matrix element in row i and column j of $(AB)^\top$ (which is the same as in row j and column i of AB) is the scalar product of row j of A and column i of B. The matrix element in row i and column j of $B^\top A^\top$ is the scalar product of row i of $B^\top$ and column j of $A^\top$, that is, of column i of B and row j of A. That is, the matrices $(AB)^\top$ and $B^\top A^\top$ have the same elements and are therefore equal.

20.17 (a) A is symmetric. (b) A is invertible with inverse B.

20.18 The 2×2-matrix P corresponding to the projection $P_a(b)$ is

$$\frac{1}{|a|^2}\begin{pmatrix} a_1^2 & a_1a_2 \\ a_1a_2 & a_2^2 \end{pmatrix}$$

Obviously, $P^\top = P$. Computing, one finds that

$$PP = \frac{1}{|a|^4}\begin{pmatrix} a_1^4 + a_1^2a_2^2 & a_1^3a_2 + a_1a_2^3 \\ a_1^3a_2 + a_1a_2^3 & a_1^2a_2^2 + a_2^4 \end{pmatrix} = \frac{1}{|a|^4}\begin{pmatrix} a_1^2|a|^2 & a_1a_2|a|^2 \\ a_1a_2|a|^2 & a_2^2|a|^2 \end{pmatrix} = P$$

20.19 $\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ rotates x the angle θ counter clockwise.

20.20 See the section about Reflection above!

20.21 (a) -4 (b) 0 (c) 10

Chapter 21

21.2 Only the rightmost one.

21.4 $(-7,2,1)$

21.5 2

21.6 $sqrt72/2$

21.7 (a) $\arccos(\frac{5}{sqrt11\sqrt{3}})$ (b) $\frac{5}{\sqrt{11}\sqrt{3}}(1,1,1)$ (c) $(1,0,-1)/\sqrt{2}$ (or $(-1,0,1)/\sqrt{2}$)

21.8 $(-1, 0, 1)$

21.9 (a) true, (b) true, (c) true

21.12

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (1)$$

21.13 Let c be any non-zero vector, not parallel to b, and consider the orthonormal basis $a_1 = b/|b|$, $a_2 = (b \times c)/|b \times c|$, $a_3 = a_1 \times a_2$, and let A be the matrix with rows a_1, a_2 and a_3, so that Ax gives the coordinates of x in the basis a_1, a_2, a_3. Let B be the first matrix in the previous problem, for rotation around the first axis. Then $C = A^\top BA$ is the desired rotation matrix. Note that A is orthogonal, so that $A^\top = A^{-1}$.

21.14

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad (2)$$

21.17 $(-1, -3, 6)$, exception.

21.18 Intersection of the two planes: $\lambda(2, -1, -1)$, intersection of two planes with the $x_1 - x_2$ plane: $(0, 0, 0)$, of course.

21.20 $r + \lambda b(a - 2P_n a)$, $(\lambda \geq 0)$, where (see figure) $b = a - 2P_n a$, $P_n a = \frac{a \cdot n}{|n|^2} n$.

Chapter 22

22.2 (a) $\frac{1}{2}$ (b) $\frac{31}{50}$ (c) $\frac{2xy}{x^2+y^2}$ if $z = x + iy$.

22.3 (a) $23 + i2$ (b) $\frac{-7-i22}{13}$ (c) $\frac{34-i22}{40}$

22.5 (a) $\sqrt{2}(\cos(45^o), \sin(45^o))$
(b) $(\cos(90^o), \sin(90^o))$
(c) $\frac{\sqrt{13}}{\sqrt{41}}(\cos(\theta - \phi), \sin(\theta - \phi))$, where $\theta = \mathrm{Arg}(2+3i)$, $\phi = \mathrm{Arg}(5+4i)$.

22.6 (a) $z_1 = (\cos(45^o), \sin(45^o))$, $z_2 = (\cos(135^o), \sin(135^o))$
(b) $z_i = (\cos(i * 45^o), \sin(i * 45^o))$, $i = 1, 2, .., 8$.
(c) $z_1 = -\frac{1}{2} + \sqrt{r}(\cos(\theta/2), \sin(\theta/2))$ and $z_2 = -\frac{1}{2} + \sqrt{r}(\cos(\theta/2 + 180^o), \sin(\theta/2 + 180^o))$, where $r = |-3/4 - i| = 5/4$ and $\theta = \mathrm{Arg}(-3/4 - i)$.
(d) hint: first solve for $w = z^2$ to find that

$$w_1 = \frac{3}{2}(1 + 2i) + \sqrt{r}(\cos(\theta/2), \sin(\theta/2))$$

and

$$w_2 = \frac{3}{2}(1+2i) + \sqrt{r}(\cos(\theta/2+180^o), \sin(\theta/2+180^o)),$$

where $r = |27/4 - 15i|$ and $\theta = \text{Arg}(27/4-15i)$. Then solve $z^2 = w_i$, $i = 1, 2$.

22.7 (a) $\{(0, y) : y \in\}$ (To see why, rewrite as $|z - (-i)| = |z - i|$)
(b) $\{(x, y) : xy = 1\}$ (Because $z^2 = (x^2 - y^2, 2xy)$)
(c) $\{(x, y) : |y| \le x\}$.

22.8 If $z = r(\cos(\theta), \sin(\theta))$ and $\zeta = \rho(\cos(\phi), \sin(\phi))$, then $z/\zeta = (r/\rho)(\cos(\theta - \phi), \sin(\theta - \phi))$.

22.10 (a) The complex plane is first rotated around the origin the angle Arg a and streched by the factor $|a|$, through the mulitiplication of z by a, then translated by the addition of b.
(b) The complex number $z = r(\cos(\theta), \sin(\theta))$ is mapped onto the complex number $r^2(\cos(2\theta), \sin(2\theta))$, that is the argument is doubled and the modulus squared.

Chapter 23

23.1 Write $x^3 = (\bar{x} + x - \bar{x})^3 = \bar{x}^3 + 3\bar{x}^2(x - \bar{x}) + 3\bar{x}(-\bar{x})^2 + (x - \bar{x})^3$. This leads to the identity $x^3 = \bar{x}^3 + 3\bar{x}^2(x - \bar{x}) + E_f(x, x)$, with the error term $E_f(x, \bar{x}) = 3\bar{x}(-\bar{x})^2 + (x - \bar{x})^3$. Note that $|E_f(x, \bar{x}| = |2\bar{x} + x|(x - \bar{x})^2$, and thus the derivative of x^3 is $3x^2$. The proof for x^4 is similar.

23.2 The error term is $E_f(x, \bar{x}) = \sqrt{x} - \sqrt{\bar{x}} - (x - \bar{x})/2\sqrt{\bar{x}} = (1/(\sqrt{x} + \sqrt{\bar{x}}) - 1/2\sqrt{\bar{x}})(x - \bar{x})$. Furthermore $1/(\sqrt{x} + \sqrt{\bar{x}}) - 1/2\sqrt{\bar{x}} = \sqrt{\bar{x}} - \sqrt{x})/(\sqrt{\bar{x}} + \sqrt{x})2\sqrt{\bar{x}} = (\bar{x} - x)/(\sqrt{\bar{x}} + \sqrt{x})^2 2\sqrt{\bar{x}}$. Collecting the results we get $E_f(x, \bar{x}) \le K_f(x, \bar{x})(x - \bar{x})^2$ with $K_f(x, \bar{x}) = 1/|(\sqrt{\bar{x}} + \sqrt{x})^2 2\sqrt{\bar{x}}| \approx 1/8\bar{x}^{3/2}$, for x close to $\bar{x}$.

23.3 We calculate the derivative of $\sqrt{x}$ at $\bar{x} = 0.5$ using the difference quotient $f'(\bar{x}) \approx f'_h(\bar{x}) = (f(\bar{x} + h) - f(\bar{x}))/h$ for $h_j = 2^{-j}$ for $j = 0, 1, \ldots, 40$ using matlab. Then we calculate the error in the numerical approximation $e_h(\bar{x}) = |f'(\bar{x}) - f'_h(\bar{x})|$. Using formula (23.27) we get $h_{optimal} = \sqrt{eps/K_f}$, where eps is the smallest number in Matlab and $K_f(x, \bar{x}) \approx 1/(8\bar{x}^{3/2})$. See the figure.

23.3 Using Taylors formula and proceeding in the same way as in Chapter 23.13 we get the following formula for the optimal choice of h: $h_{optimal} = (eps/K_f)^{1/3}$, with $K_f = f'''(\bar{x})/6$. For $f(x) = \sqrt{(x)}$ we show the error in the difference quotient as a function of h as well as the the predicted optimal h (vertical dashed line).

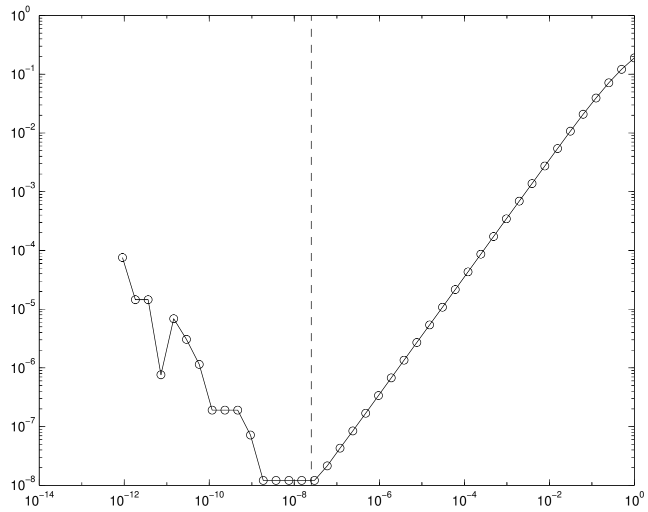


Figure 17: The error e_h in the numerical derivative as a function of h and the predicted optimal choice of h marked by a vertical dashed line. Note that this is a log-log plot! (Problem 23.3)

23.5 For simplicity, compute the derivative at $\overline{x} = 1$. Then the relative error for a specific choice of $h = x - \overline{x}$ is $e_n(h) = \frac{\frac{(1+h)^n - 1}{h} - n}{n}$. The relative errors for a few different choices of n are plotted as function of h in the figure below. For $n = 1$ one should choose a large value of h since the linearization error is zero and we only need to worry about round-off/computational error. For larger n there is an optimal value of h.

23.6 Perhaps the correct answer to this question is no, since we have not yet defined $\sin(x)$ and $\cos(x)$, but we still may find the correct answer. Two alternatives: (i) Realize geometrically that $\sin(x) \approx x$ for small $|x|$. Then use the relations $\sin(x) - \sin(\overline{x}) = 2\sin\left(\frac{x-\overline{x}}{2}\right)\cos\left(\frac{x+\overline{x}}{2}\right)$ and $\cos(x) - \cos(\overline{x}) = -2\sin\left(\frac{x-\overline{x}}{2}\right)\sin\left(\frac{x+\overline{x}}{2}\right)$. (ii) The second alternative is to realize directly geometrically that the derivative of $\sin(x)$ is $\cos(x)$ and the derivative of $\cos(x)$ is $-\sin(x)$.

23.7 Use Theorem 23.1 to get a lower bound for L and then show that the function is really Lipschitz continuous with this L.

23.8 Use the fact that $f(x_i) = f(0) + (x-0)f'(0) + E_f(x_i, 0)$ and $g(x_i) = g(0) + (x-0)g'(0) + E_g(x_i, 0)$, which gives $f(x_i) = xf'(0) + E_f(x_i, 0)$ for f and the same for g. Divide by x and realize that the limit is $f'(0)/g'(0)$.

23.9 This problem should perhaps be in the next chapter?

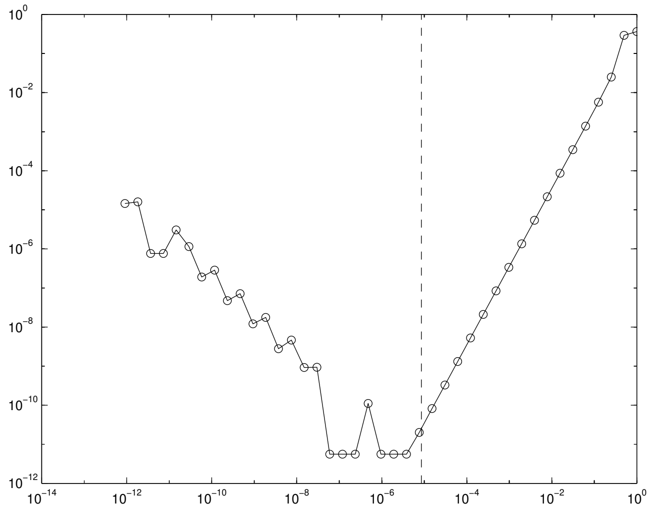


Figure 18: The error e_h in the numerical derivative as a function of h and the predicted optimal choice of h marked by a vertical dashed line. Note that this is a log-log plot! (Problem 23.4)

Generalize l'Hopital's rule to $x_i \to \overline{x}$ and compute the derivatives at $\overline{x} = 1$. The limits are $1/2$ and r.

Chapter 24

24.1 The rules for differentiating x^r, the quotient rule, and the chain rule gives:

$$D\left(\sqrt{x^{11}+\sqrt{\frac{x^{111}}{x^{-1.1}+x^{1.1}}}}\right) = \frac{1}{2\sqrt{x^{11}+\sqrt{\frac{x^{111}}{x^{-1.1}+x^{1.1}}}}}\cdot\left(11x^{10}+\right.$$

$$\left.+\frac{1}{2\sqrt{\frac{x^{111}}{x^{-1.1}+x^{1.1}}}}\cdot\frac{111x^{110}(x^{-1.1}+x^{1.1})-x^{111}(-1.1x^{-2.1}+1.1x^{0.1})}{(x^{-1.1}+x^{1.1})^2}\right)$$

24.2 $\frac{\partial f}{\partial x_1} = \frac{\partial}{\partial x_1}(x_1^2+x_2^4)\partial x_2(x_1^2+x_2^4) = 4x_2^3$

24.3 We plot q_n for $n = 2^1, 2^2, ..., 2^{15}$: By increasing n, we find that q_n converges to $0.6931... = \ln 2$. Now, in the Chapter A Very Short Course in Calculus, we saw that $\lim_{\to\infty}(1+\frac{1}{n})^n = e$. We see the connection to $\lim_{n\to\infty}(1+\frac{q_n}{n})^n = 2$, by noting that $\lim_{n\to\infty} q_n = D2^x(0)$ and $De^x(0) = 1$

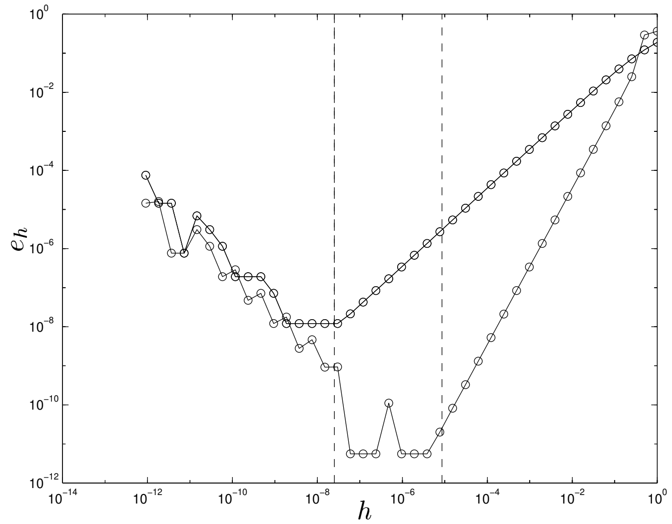


Figure 19: Comparision between the results in Problem 23.3 and 23.4. Note the improvement using the symmetric difference formula.(Problem 23.4)

24.4 Let $f(x) = 2^x$ and suppose that we know $f'(0)$ (see Problem 24.3) We have $2^x = 2^{x-\bar{x}}2^{\bar{x}}$, i.e., $f(x) = f(x-\bar{x})f(\bar{x})$. The chain rule gives $f'(x) = f'(x-\bar{x})f(\bar{x})$, so that $f'(\bar{x}) = f'(0)f(\bar{x})$.

24.5 (i) $a + b = \frac{1}{2}$ (match the two pieces at $x = 1$)
(ii) $a = -2$, $b = \frac{5}{2}$ (match also the left- and right-handderivatives at $x = 1$)

Chapter 25

25.4 $\bar{x} = \{-3.3027, -1.6180, 0.3027, 0.6180, 1\}$, *Note that the answers have not been rounded off.*

25.5 $\bar{x} = 3.0608$

25.5 Probably an error in the assignment. $x_0 = 1/\sqrt{3}$ is more interesting.

25.8 (a) E.g. $x_i - \bar{x} \approx \frac{1}{1-g'(x_i)}(x_i - g(x_i))$

Chapter 27

27.1 (a) $\int 2x/(1+x^2)dx = -1/(1+x^2) + c$,
(b) $\int (1+x)^{-99}dx = -(1+x)^{-98}/98 + c$,
(c) $\int 2(1+x^3)3x^2/(1+(1+x^3)^2)dx = -1/(1+(1+x^3)^2) + c$,
where c is an arbitrary constant.

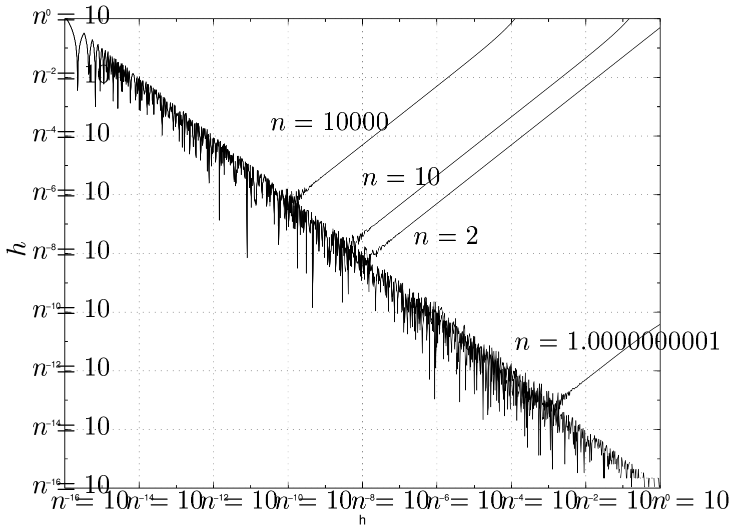


Figure 20: Relative errors for the numerical derivative as function of h. (Problem 23.5)

27.2 The area is given by the integral $\int_1^2 (1+x)^2 dx = [-(1+x)^{-1}]_1^2 = 1/6$.

27.3 Note that the velocity is the time derivative of the distance traveled, i.e., $v(t) = u'(t)$. Thus the distance is given by $u(t) = \int_0^{10} v(t)dt = \int_0^{10} t^{3/2} dt = [2t^{5/2}/5]_0^{10} = 200\sqrt{10}$.

27.7 i. $u(x) = 1$,

ii. $u(x) = x + 1$,

iii. $u(x) = \frac{x^{r+1}}{r+1} + 1$.

27.8 i. $u(x) = x + 1$,

ii. $u(x) = \frac{x^2}{2} + x + 1$,

iii. $u(x) = \frac{x^{r+2}}{(r+1)(r+2)} + x + 1$.

We need two conditions since all functions of the form $Ax+B$, where A and B are two arbitrary constants, will disappear when we take the second derivative.

27.9 The solution is

$$u(x) = \begin{cases} x+1, & x \in [0,1), \\ 2x, & x \in [1,2]. \end{cases}$$

From the graph of the function f it is easy to see that it is not Lipschitz continuous. Alternatively, assume that f is Lipschitz continuous and choose $x - y = 1/2L$ for $x < 1$, $y > 1$. Then $|f(x) - f(y)| \leq L|x-y| = 1/2$ from the Lipschitz condition and at the same time

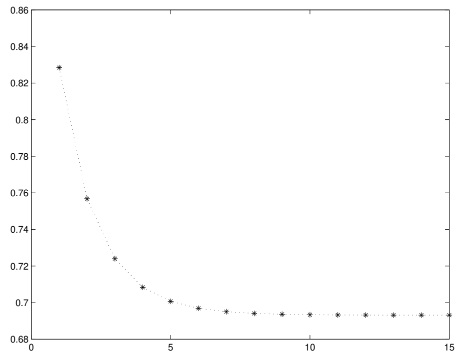


Figure 21: Problem 24.3.

we have $|f(x) - f(y)| = |1 - 2| = 1$. This is a contradiction and thus f is not Lipschitz continuous. To see that u is Lipschitz continuous, note that for $x < 1$ we have $x + 1 > 2x$ so that for $x < 1$, $y > 1$ (the reverse case is treated similarly and it is straightforward to prove that u is Lipschitz continuous on $[0, 1]$ and on $[1, 2]$) we have $|f(x)-f(y)| = |1+x-2y| = 2y-(1+x) \leq 2y-2x = 2(x-y) = 2|x-y|$. Thus u is Lipschitz continous on $[0, 2]$ with $L = 2$. (It may be simpler to see that u is Lipschitz continuous on $[0, 2]$ with Lipschitz constant $L = 3$. Although that is true, we managed here to show a better result.)

27.10 $t = \frac{2}{c}\left(hn_g + \int_0^R n_w(r)\ dr\right)$.

27.11 If $f = g$ on $[0, 1]$ then clearly $\int_0^1 |f(x) - g(x)|\ dx = \int_0^1 0\ dx = 0$. To show the converse, that $\int_0^1 |f(x) - g(x)|\ dx = 0$ implies $f = g$ on $[0, 1]$, assume that $\int_0^1 |f(x) - g(x)|\ dx = 0$ and note that if $f \neq g$ at some point $x \in [0, 1]$ then we must have $f \neq g$ in some interval I_x containing x, since f and g are Lipschitz continous (see below). This means that $\int_0^1 |f(x) - g(x)|\ dx \geq \int_{I_x} |f(x) - g(x)|\ dx > 0$. This is a contradiction and thus we cannot have $f(x) \neq g(x)$ at any point in $[0, 1]$. Hence $f = g$ in $[0, 1]$.

Now, to see that $f(x) \neq g(x)$ at some point x implies that $f \neq g$ in some interval containing x if f and g are Lipschitz continous, let $h = f - g$ and note that if $h(x) \neq 0$ then $|h(y)| = |h(x) + h(y) - h(x)| \geq |h(x)| - |h(y) - h(x)| \geq |h(x)| - L_{f-g}|x - y| > 0$ if $x - y$ is sufficiently small (and also $y \in [0, 1]$). (Note that $f - g$ is also Lipschitz continous.) Thus for y in some small interval containing x we have $|f - g| > 0$ which is the same as $f \neq g$.

If $\int_0^1 |f(x) - g(x)|\ dx$ is replaced by $\int_0^1 (f(x) - g(x))\ dx$ then this integral is still zero if $f = g$ on $[0, 1]$ but we may no longer conclude that $f = g$ if the integral is zero, since we can have for e.g. $f = x$ and $g = 1 - x$ that $\int_0^1 (f(x) - g(x))\, dx = 0$ while $f \neq g$.

Chapter 28

28.1 (a) $a/2 + b/3$

(b) 1

(c) 2

(d) $-2a$ for $a \leq -1$, $1 + a^2$ for $-1 < a < 1$, and $2a$ for $1 \leq a$.

(e) $((1-a)^1 1 + (1+a)^1 1)/11$.

28.2 (a) $x^3/3$

(b) $x^4/4$

(c) $x^4/4$

(d) $x^3/3 - x$

28.3 Integration by parts gives $1/1003002$.

28.4 (a) 0

(b) $(f(7) - f(0))/7$

(c) $(f(124) - f(165))/17$

28.5 $1/22$.

28.6 (a) x

(b) $3 + 6(x-1) + 4(x-1)^2 + (x-1)^3$

(c) Take $f(x) = \sqrt{x+1}$ instead. Then the Taylor polynomial is $1 + x/2 - x^2/8 + 3x^3/48 - 5x^4/128 + \ldots$

28.7 The Taylor series is $f(x) = r(x-1) + r(r-1)(x-1)^2/2 + r(r-1)(r-2)(x-1)^3/6 + \ldots$, and the limit is r.

28.10 The first formula, $\int\, dy = y$, states that the total length is the sum of its parts. This thus gives us that y is a primitive function of 1.

The second formula, $\int y\; dy = \frac{y^2}{2}$, states that the area of a right triangle, the one with corners at $(0,0)$, $(y,0)$ and $(0,y)$ is $\frac{y^2}{2}$. This thus gives us that $\frac{y^2}{2}$ is a primitive function of y.

The third formula, $\int x\; dy = xy - \int y\; dx$, states that the area of a rectangle with sides of lengths x and y is xy, which is equal to the sum of the area $\int y\; dx$ below the curve $y = y(x)$ and the area to the left of the curve $x = x(y)$ for a curve going from $(0,0)$ to (x,y). This thus gives us the formula for partial integration,

$$\int x \frac{dy}{dx}\; dx = xy - \int y\; dx,$$

since $dy = \frac{dy}{dx} dx$.

28.11 Use the mean value theorem for integrals on the last term in Taylor's theorem:

$$\begin{aligned}\int_{\bar x}^{x} \tfrac{(x-y)^n}{n!} u^{(n+1)}(y)\ dy &= u^{(n+1)}(\hat x) \int_{\bar x}^{x} \tfrac{(x-y)^n}{n!}\ dy \\ &= \tfrac{u^{(n+1)}(\hat x)}{(n+1)!}(x-\bar x)^{n+1}.\end{aligned}$$

28.12 Integrating by parts we get

$$\begin{aligned}\int_0^{\bar y} f(y)\, dy &= \int_0^{\bar x} x \tfrac{dy}{dx}\, dx \\ &= \int_0^{\bar x} x(f^{-1})'(x)\, dx \\ &= \bar x f^{-1}(\bar x) - \int_0^{\bar x} f^{-1}(x)\, dx \\ &= \bar x \bar y - \int_0^{\bar x} f^{-1}(x)\, dx.\end{aligned}$$

This is the same as Leibniz' formula for integration by parts with a bit more explicit notation for the limits.

28.13 For $x < y$ we have $|F(x) - F(y)| = |\int_x^y f(x)\, dx| \le \int_x^y |f(x)|\, dx \le \int_x^y L_F\, dx = L_F \int_x^y dx = L_F|x-y|$, if $x, y \in [0, a]$. The same thing will hold with x and y interchanged if $x > y$. Thus F is Lipschitz continuous on $[0, a]$ with Lipschitz constant L_F.

28.14 Since the integral is Lipschitz even if the function is not, if the function is bounded as we had in the previous exercise. It is better to be Lipschitz than bounded, since a Lipschitz continuous function is also bounded (on any finite interval in the domain of Lipschitz continuity).

28.15 Integrate by parts. The functions f and φ should be n times differentiable and $\frac{d^n\phi}{dx}$, $n = 0, 1, \ldots, n-1$ or $\frac{d^n f}{dx}$, $n = 0, 1, \ldots, n-1$ should be zero at the endpoints of the interval.

28.16 Notice that $0 \le (\bar u - (\bar u, \bar v)\bar v, \bar u - (\bar u, \bar v)\bar v) = 1 - 2(\bar u, \bar v)(\bar u, \bar v) + (\bar u, \bar v)^2 = 1 - |(\bar u, \bar v)|^2$. Thus $|(\bar u, \bar v)| \le 1$, which gives $|(u, v)| \le \|u\|\|v\|$.

28.17 Take $\bar x$ as one of the endpoints of the interval I. Notice that $v(x) = v(\bar x) + \int_{\bar x}^x v'(y)\, dy = \int_{\bar x}^x v'(y)\, dy$. Thus $|v(x)| \le \int_I |v'(y)|\, dy = \int_I 1 \cdot |v'(y)|\, dy \le \|1\|_{L^2(I)}\|v'\|_{L^2(I)}$ by Cauchy's inequality. This gives $\|v\|_{L^2(I)} = \sqrt{\int_I |v(x)|^2\, dx} \le \sqrt{\int_I \|1\|^2_{L^2(I)}\|v'\|^2_{L^2(I)}\, dx} = \|1\|_{L^2(I)}\|v'\|_{L^2(I)}\sqrt{\int_I dx} = \|1\|^2_{L^2(I)}\|v'\|_{L^2(I)} = \int_I dx \|v'\|_{L^2(I)} = C_I\|v'\|_{L^2(I)}$, where the constant C_I is the length of the interval.

28.18 Since $C_I = 1$, we want to show that $\|v\|^2_L(I) \le \|v'\|^2_L(I)$.

a) $\|v\|^2_L(I) = \frac{1}{\sqrt{30}} \le \|v'\|^2_L(I) = \frac{1}{\sqrt{3}}$,

b) $\|v\|^2_L(I) = \frac{1}{\sqrt{105}} \le \|v'\|^2_L(I) = \sqrt{\frac{2}{15}}$,

c) $\|v\|^2_L(I) = \frac{1}{\sqrt{105}} \le \|v'\|^2_L(I) = \sqrt{\frac{2}{15}}$.

Chapter 29

29.1 Note that $\log(4) = \int_1^4 x^{-1}dx \geq 1 \times 1/2 + 2 \times 1/4 = 1$, and $\log(2) = \int_1^2 x^{-1}dx \geq 1 \times 1/2 = 1$.

29.2 $\log(2^n) = n\log(2) \to \infty$ as $n \to \infty$. $\log(2^{-n}) = -n\log(2) \to -\infty$ as $n \to \infty$.

29.3 The correct change of variables should be $z = y/a$, which leads to $\int_a^{ab} y^{-1}dy = \int_1^b z^{-1}dz = \log(b)$.

29.4 Note that $\log'(1+x) = 1/(1+x) \leq 1 = x'$ and the inequality holds for $x = 0$ and thus it holds for $x > 0$ since x grows faster than $\log(1+x)$. The other case is similar.

29.5 By the Meanvalue theorem $\log(1+x) = \int_1^{1+x} z^{-1}dz = x\xi^{-1}$, with $\xi \in (1, x)$, and $x\xi^{-1} \leq x$ since $\xi^{-1} \leq 1$. The other case is similar. Estimating the area we have, for $x > 1$, we have $\log(1+x) = \int_1^{1+x} z^{-1}dz \leq x$ and for $-1 < x < 1$ we get $\log(1+x) = -\log(1/(1+x)) = -\int_1^{1/(1+x)} z^{-1}dz \leq x$.

29.6 $\log(a) - \log(b) = \log(a) + \log(1/b) = \log(a) + \log(1/b) = \log(a/b)$.

29.7 $\sum_{k=1}^n (-1)^{n-1}\frac{(x-1)^n}{n}$.

29.8 $\frac{1}{2}\log\frac{|x-1|}{|x+1|}$.

29.9 $\log(x^r) = \log(x^{p/q}) = p\log(x^{1/q}) = \frac{p}{q}q\log(x^{1/q}) = \frac{p}{q}\log(x) = r\log(x)$.

29.10 Plot $u(x) = \frac{x^{-a+1}}{-a+1} - \frac{1}{-a+1}$ for different values of $a \neq 1$ but close to 1. Note that u tends to infinity when x tends to infinity if $a < 1$. Since we know that the solution is $u(x) = \log(x)$ for $a = 1$, the solution tends to infinity if $a \leq 1$.

29.11 a) $x = \left(\frac{5}{3}\right)^{2/3}$,
b) $x = 7/3$,
c) $x = 7^{1/4}$.

29.12 a) $\frac{3x^2+6}{x^3+6x}$,
b) $\frac{1}{x\log(x)}$,
c) $\frac{1+2x}{x+x^2}$,
d) $-\frac{1}{x}$,
e) $\log(x)$.

Chapter 30

30.1 (a) For endpoint: $Q_h \leq 2h$, and midpoint $Q_h = 0$.
(b) For endpoint: $Q_h \leq \frac{h}{2}\sum_{j=1}^N h3x_j^2 \approx 4h$ for small h (why ?) and for midpoint: $Q_h \leq \frac{h^2}{8}\sum_{j=1}^N h2x_j \approx \frac{h^2}{2}$ for small h.

(c) For endpoint: $Q_h \leq \frac{h}{2}\sum_{j=0}^{N-1} he^{-x_j} \approx \frac{h}{2}(1-e^{-2})$, and for midpoint $Q_h \leq \frac{h}{24}\sum_{j=0}^{N-1} he^{-x_j} \approx \frac{h}{24}(1-e^{-2})$.

30.4 Error estimate for the trapezoidal rule $Q_h \leq \frac{1}{12}\sum_{j=1}^{N}(\max_{y\in I_j}|f''(y|h_j^2)h_j$

30.6 The Gauss points on $I=[-1,1]$ are $\hat{x}_1 = -1/\sqrt{3}$ and $\hat{x}_2 = 1/\sqrt{3}$. The resulting quadrature rule is exact for polynomials of order three.

30.7 Error in the formulation of the problem ! Square the result should be multiply the result by four. You will find an approximation of π.

Chapter 31

31.1 With $U^n(x_i^n) = U^n(x_{i-1}^n) \pm h_n U^n(x_i^n)$, we get $U^n(x_i^n) = \frac{1}{(1-\pm h_n)^n}$. Proceed in the same way as in the original construction of the exponential function.

31.2 **Note! Misprint. Should be $x > -n$.**
Use that that $\log(1+y) \leq y$ with $y = x/n > -1$.

31.3 Let $v(x) = 1/u(x)$. Then $v' = -\frac{1}{u^2}u = -1/u = -v$. Since also $v(0) = 1/u(0) = 1$, v solves the equation.

31.4 The first expression is the construction of the exponential at level n, the second the one at level $n+1$. Since the increment from each point is the function value itself in that point, taking two steps over two intervals gives a larger value than taking one step over the whole interval. Convince yourself of this by drawing a figure. If you are still not convinced, then let $y = \frac{x}{2^n}$ and notice that

$$(1+\frac{1}{x^{2n}})^{2n} = (1+y)^{2n} \leq (1+y+\frac{y^2}{4})^{2n} = (1+y/2)^{2n+1} = (1+\frac{1}{x^{2n+1}})^{2n+1}.$$

31.5 The crucial step is to show that $\binom{n}{k}\frac{1}{n^k} \leq \binom{n+1}{k}\frac{1}{(n+1)^k}$. Show this by writing down the expressions for the binomial coefficients, boiling it down to just k factors in both expressions, divide every factor by n in the left-hand side and by $n+1$ in the right-hand side. Simplify and conclude that the left-hand side is smaller than the right-hand side.

31.6 If the interest over the year a is divided into parts, so that at N times during the year, an interest of a/N is added, then the capital u is changed from year n to year $n+1$ as

$$u_{n+1} = (1+\frac{a}{N})^N u_n,$$

so that the effective annual interest is $\bar{a} = (1+\frac{a}{N})^N$. For large values of N we have $\bar{a} \approx \exp(a)$, which in turn is approximately equal to a if the interest is small. (As it usually is ...)

31.7 Using the Taylor expansion of $\exp(\lambda x)$, we see that as λ tends to zero, the solution tends to $u(x) = x^2/2$, as it should, since for $\lambda = 0$ the equation is $u' = x$ with the initial condition $u(0) = 0$.

31.8 **This exercise should be removed.**
See exercise 31.11.

31.9 $\frac{d}{dx}x^r = \frac{d}{dx}\exp(\log(x)r) = \exp(\log(x)r)r/x = x^r r/x = rx^{r-1}$.

31.10 **Note! Misprint. Should be $\lambda(x) = x^r$.**
$u(x) = u_0 \exp(\frac{x^{r+1}}{r+1})$.

31.11 $\exp(a)\exp(b) = \exp(\log(\exp(a)\exp(b))) = \exp(\log(\exp(a))+\log(\exp(b))) = \exp(a+b)$ and $(\exp(a))^\alpha = \exp(\log((\exp(a))^\alpha)) = \exp(\alpha\log(\exp(a))) = \exp(\alpha a)$.

31.12 Let u be the solution of $u' = u$ on $(0,1]$, $u(0) = 1$. Then $v(x) = u(x-1)$ solves $v' = v$ on $(1,2]$, $v(1) = u(0)$, since $v'(x) = u'(x-1) = u(x-1) = v(x)$ and $v(1) = u(0)$. Now, $w(x) = u(1)v(x)$ solves $w' = w$ on $(1,2]$ and $w(1) = u(1)$, so

$$\bar{u}(x) = \begin{cases} u(x), & x \in [0,1], \\ w(x), & x \in (1,2], \end{cases}$$

is a continuously differentiable solution to $\bar{u}' = \bar{u}$ in $(0,2]$ with $\bar{u}(0) = 1$. Continuing this way we can construct a solution for all $x \geq 0$. To construct a solution for $x < 0$, consider $1/u(-x)$.

31.13

$$\begin{aligned} \exp(x) &= 1 + x + \tfrac{x^2}{2} + \tfrac{x^3}{6} + \ldots + \tfrac{x^n}{n!} + \tfrac{x^{n+1}}{(n+1)!}\exp(\hat{x}(x)) \\ &= \textstyle\sum_{i=0}^{n} \frac{x^i}{i!} + \frac{x^{n+1}}{(n+1)}\exp(\hat{x}(x)), \end{aligned}$$

for some $\hat{x}(x) \in (0,x)$.

31.14 a) $-\frac{1}{2}\exp(-x^{)}$,
b) $-\frac{1}{2}\left(x^2+1\right)\exp(-x^2)$.

31.15 a) $\log(a)a^x$,
b) $\exp(x+1)$,
c) $\left(1+2x^2\right)\exp(x^2)$,
d) $\left(3x^2+2x^4\right)\exp(x^2)$,
e) $-2x\exp(-x^2)$.

31.16 a) $\frac{a-1}{\log(a)}$,
b) $\exp(2) - \exp(1)$,
c) $\frac{\exp(1)-1}{2}$,
d) $\frac{1}{2}$,
e) One cannot express the primitive function of $\exp(-x^2)$ in simple terms of e.g. exponentials, polynomials or logarithms.

31.17 The number is π!

31.18 For $x > -1$, we have $\log(1+x) \le x$, which gives $1 + x \le \exp(x)$ for $x > -1$. For $x <= -1$ we have $1 + x \le 0 \le \exp(x)$.

31.19 **Note! Misprint. Should be** $\exp(x) \ge 1 + x$**.**
For $n = 0$ the left-hand side is $\mathrm{e}^x f(x)$ and the right-hand side is $\mathrm{e}^x \cdot 1 \cdot f(x) = \mathrm{e}^x f(x)$. The induction step:

$$\begin{aligned}
\tfrac{d^{n+1}}{dx^n}\left(\mathrm{e}^x f(x)\right) &= \tfrac{d}{dx}\tfrac{d^n}{dx^n}\left(\mathrm{e}^x f(x)\right) \\
&= \tfrac{d}{dx}\left(\mathrm{e}^x\left(1+\tfrac{d}{dx}\right)^n f(x)\right) \\
&= \mathrm{e}^x\left(1+\tfrac{d}{dx}\right)^n f(x) + \mathrm{e}^x \tfrac{d}{dx}\left(1+\tfrac{d}{dx}\right)^n f(x) \\
&= \mathrm{e}^x\left(\left(1+\tfrac{d}{dx}\right)^n + \tfrac{d}{dx}\left(1+\tfrac{d}{dx}\right)^n\right) f(x) \\
&= \mathrm{e}^x\left(1+\tfrac{d}{dx}\right)\left(1+\tfrac{d}{dx}\right)^n f(x) \\
&= \mathrm{e}^x\left(1+\tfrac{d}{dx}\right)^{n+1} f(x).
\end{aligned}$$

Chapter 32

32.1–4 Insert the given expressions into the differential equations and verify that the equations and boundary conditions are satisfied. **Misprint: Problem 32.3** $r\cos(\sqrt{k}(t-\alpha)$ should be $r\cos(\sqrt{k}(t+\alpha)$.

32.5 $u(x) = -\cos(x) + 2/3 - 1/\sqrt{2}$.

32.6 let $v(x) = x$ and $u(x) = \sin(x)$. Note that $u'(x)^2 + u(x)^2 = 1$ for all x. In particular $|u'(x)| < 1 = v'(x)$ for all x and since $v(0) = u(0)$ the first inequality follows. The second inequality follows in the same way by observing that $tan(x))' = 1/\cos(x)^2 > 1$ for $x \in (0, \pi/2)$.

32.7 $\sin(x)/x = (x - cos(y)x^2/2)/x = 1 - cos(y)x/2 \to 1$ as $x \to 0$ (for some $y \in [0, x]$).

32.10 (a) $5\cos 1 - 3\sin 1$.
(b) $e(\sin 1 - \cos 1 + 1)/2$.
(b) $\sin 1 + 2\cos 1$.

32.11 $\pi = 4\arctan(1) = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \dots\right)$. Use a few terms to get an approximation of π. One term gives $\pi \approx 4$, which is a crude estimate of π. Using more terms gives a better estimate. However, this method we will need a very large number of terms to get a reasonable accurate result, since the terms converge to zero quite slowly. For five correct decimals you will need about 10^6 terms!

32.12 Take the tan of both sides and use formula (32.20). The suitable numbers are $a, b \in (0, 1)$ such that $b = \frac{1-a}{1+a}$. Minimizing the largest of a and b gives $a = b = \sqrt{2} - 1$, which is not a rational number. Choose a and b rational, but close to this value.

32.13 The Taylor expansion of $\arctan(x)$ converges faster for small values of $|x|$ — for $|x| > 1$ it doesn't converge at all — since in the expansion we take powers of the argument.

With $a = 1/2$ and $b = 1/3$, we get five correct decimals in less than ten iterations, which is surely an improvement. The formula suggested in the exercise gives even better convergence, since the arguments are even smaller.

32.14 a) For $x \in [-1, 1]$, we have $\sin(\arcsin(x)) = x$. Thus $\sin(\arcsin(-x)) = -x = -\sin(\arcsin(x) = \sin(-\arcsin(x))$ and, since $\arcsin(x) \in [-\pi/2, \pi/2]$ and sin is increasing on this interval, we may conclude that $\arcsin(-x) = -\arcsin(x)$.

b), c), d), e) and f) are treated in a similar way.

32.15
i. $\frac{\pi}{8}$,
ii. $\frac{\pi}{4}$,
iii. $\frac{\pi}{3}$,
iv. $\frac{1}{3}$,
v. $\frac{24}{25}$,
vi. $\frac{3\pi}{4}$.

32.16
i. $\sqrt{\frac{1}{2}(\sqrt{2}-1)}$,
ii. $\frac{\pi}{4}(\sqrt{3}-1)$.

32.17
i. $\frac{\frac{1}{2\sqrt{x}}-5x^4}{1+x-2x^{11/2}+x^{10}}$,
ii. The derivative does not exist ! Why ? Hint: check the domain of definition of arcsinus.
iii. $\frac{2x}{(1-x^4)^{3/2}}$,
iv. $-\frac{1}{2\sqrt{x}(1+x)(\arctan(\sqrt{x}))^2}$.

32.18 For computing $x = \arcsin(y)$ note that it is the solution of $\sin(x) = y$, i.e. $f(x) = 0$ for $f(x) = \sin(x) - y$. Solve this equation in the standard way, i.e. by fixed point iteration, e.g. Newton's method.

32.19 Do the same as when calculating the derivatives of arcsin and arccos, and use that $\cosh^2(x) - \sinh^2(x) = 1$.

32.20 With the substitution $x = \sin(y)$ we get $\int_{-1}^{1} \sqrt{1-x^2}\, dx = \int_{-\pi/2}^{\pi/2} \cos^2(y) dy = \pi/2$.

Chapter 33

33.1 (a) $-x\frac{1}{2}\cos(2x) + \frac{1}{4}\sin(2x)$
(b) $x^2 \sin(x) + 2x\cos(x) - 2\sin(x)$
(c) $-\frac{1}{2}\exp(-2x) - \frac{1}{4}\exp(-2x) + \frac{1}{4}$

33.2 (a) $\frac{1}{2}x^2\log(x) - \frac{1}{4}x^2 + \frac{1}{4}$
(b) $x\log(x) - x + 1$
(c) $x\arctan(x) - \frac{1}{2}\log(1+x^2)$
(d) $\frac{4}{5}(\exp(-x)(\frac{1}{2}\sin(2x) - \frac{1}{4}\cos(2x)) + \frac{1}{4})$

33.3 (a) $\frac{1}{2}\log(1+x^2)$
(b) $\log(1+e^x) - \log(2)$

33.4 (a) $\frac{1}{2}\exp(x^2) - \frac{1}{2}$, with substitution $z = y^2$.
(b) $\frac{2}{5}((x-1)^{5/2} - (-1)^{5/2}) + \frac{2}{3}((x-1)^{3/2} - (-1)^{3/2})$, with substitution $z = y - 1$.
(c) $\frac{1}{3}((-\cos(x))^3 - (-1)^3)$

33.5 (a) $-\frac{1}{3}\log(|x+1|) + \frac{1}{3}(\log(|x-2|) - \log(|-2|))$, for $-1 < x < 2$.
(b) $\frac{1}{2}x^2 - 2x - \frac{1}{4}\log(|x-1|) + \frac{29}{4}(\log(|x+3|) - \log(|3|))$, for $-3 < x < 1$.
(c) $\frac{1}{4}(\arctan(\frac{x+1}{2}) - \arctan(\frac{1}{2}))$
(d) $\frac{1}{8}\log(|x-1| - \frac{1}{8}...$

33.7 (a) $2(\pi\sin(\pi) + \cos(\pi)) = -2$
(b) π
(c) 0
(d) 0

Chapter 36

36.1 (a) $u(x) = (x + u_0^{-1})^{-1}$
(b) $u(x) = (x + s\sqrt{u_0})^2/4$
(c) $u(x) = \exp(\exp(x + \log(\log(u_0))))$
(d) $u(x) = \tan(x + \arctan(u_0))$
(e) $u(x) = \arctan(\exp(x + \ln|\tan(u_0/2)|))$
(f) $u(x) = -1 + \sqrt{1 + 2x + 2u_0 + u_0^2}$.

36.4 The velocity satisfies the equation $v' = g - bv^2$ with $v(0) = 0$ and b a positive constant. The solution is

$$v = \frac{\sqrt{g}(e^{2gt} - 1)}{\sqrt{b}(e^{2gt} + 1)},$$

check units and compute the limit of the velocity as $t \to \infty$.

36.4 $t = e^5 + 1$.

Chapter 37

37.1 See figures 22 and 23 for the numerical solution.
The extended system is not separable.

37.2 Note that $\int_0^T \frac{\dot{x}}{x}\,dt = \int_{x(0)}^{x(T)} \frac{1}{x}\,dx = \log\left(\frac{x(T)}{x(0)}\right) = 0$, since $x(0) = x(T)$. Thus $\int_0^T (a - by)\,dt = 0$, which gives $\hat{y} = \frac{a}{b}$. Same for $\hat{x}$. Adding dissipative terms $-\epsilon x$ and $-\epsilon y$ gives $\hat{x} = \frac{c-\epsilon}{d}$, $\hat{y} = \frac{a-\epsilon}{b}$.

37.3 See figure 24.

37.4 **Note! Misprint. Should read "Show that for $b = 1$ and $w = 0$, the economy oscillates for $a > 1$." End of exercise, i.e. you can ignore the rest of this exercise, from "Show that there ... "**

To find the equilibrium state, set $\dot{u} = \dot{v} = 0$.

To see that the economy oscillates for $a > 1$, sketch the phase-plane curves or write the system as $\bar{u}' = A\bar{u}$ for $\bar{u} = (u, v)$ and some matrix A. Determine a so that A has imaginary eigenvalues.

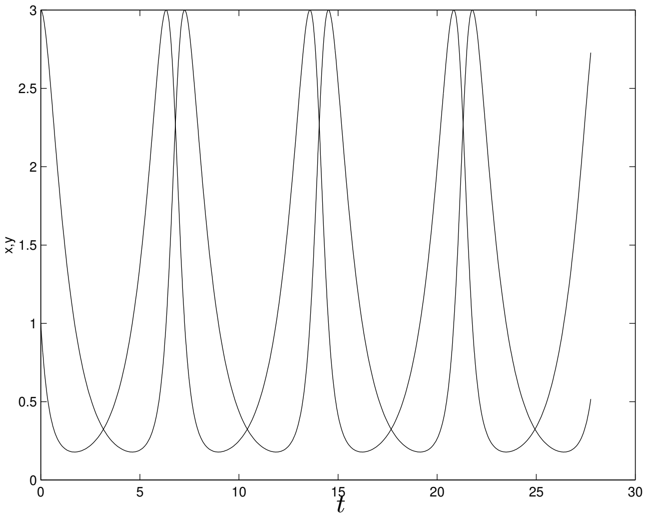


Figure 22: A typical solution to the standard Volterra-Lotke equations, with $a = b = c = d = 1$. (Problem 37.1)

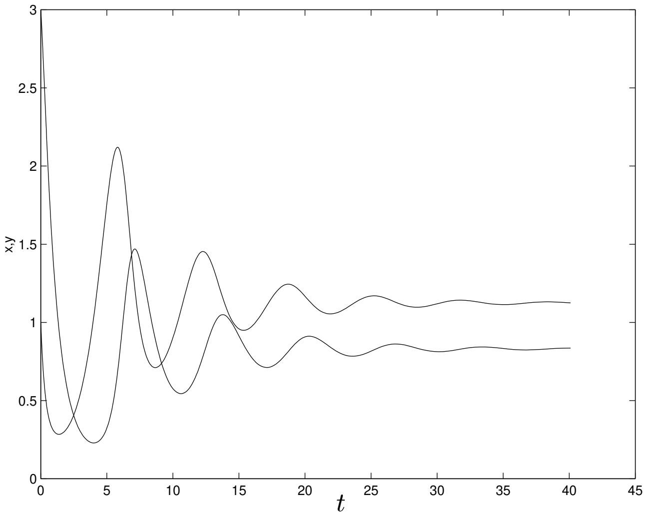


Figure 23: A typical solution to the extended Volterra-Lotke equations, with $a = b = c = d = e = f = 1$. (Problem 37.1)

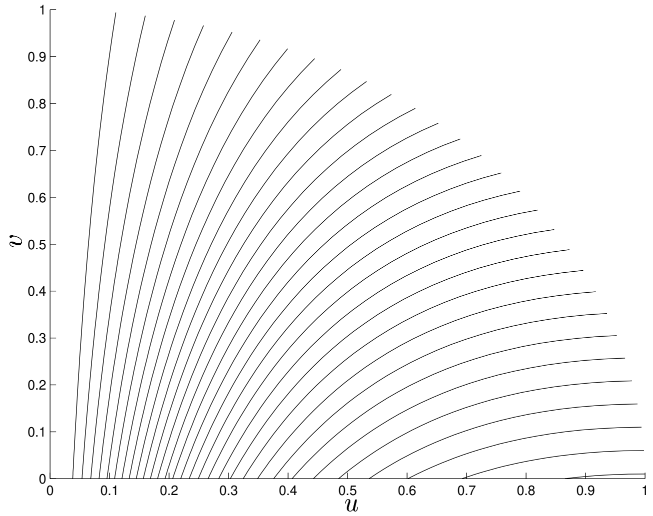


Figure 24: Phase-plane curves for the solution for a number of different initial values for u and v. The equilibrium is at $v = 0$, i.e. no infectives. The number of remaining suscebtibles depends on the initial values. (Problem 37.3)

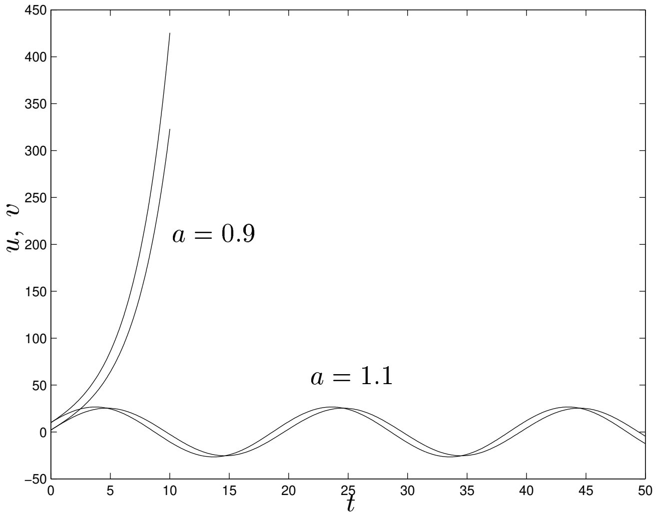


Figure 25: The solution for $b = 1$ and $w = 0$ for different values of a. For $a > 0$ the solution oscillates.