PDE Project Course

1. Adaptive finite element methods

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Lecture plan

- Introduction to FEM
- FEM for Poisson's equation
- Adaptivity for Poisson's equation
- FEM for $\dot{u} = f$
- Adaptivity for $\dot{u} = f$

Introduction to FEM

A method for solving PDEs

The finite element method (FEM), also known as Galerkin's method, is a general method for solving PDEs (or ODEs) of the form

$$A(u) = f,$$

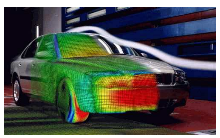
where A is a differential operator, f is a given force term and u is the solution.

Solving PDEs

- Analytic solutions can be obtained only for simple geometries in special cases:

$$\boxed{-\Delta u = 0}$$

- Using the computer, we can obtain solutions to general problems with complex geometries:



$$\begin{aligned} \dot{u} + u \cdot \nabla u - \nu \Delta u + \nabla p &= f \\ \nabla \cdot u &= 0 \end{aligned}$$

The finite element method

Find an approximate solution U of the form

$$U(x) = \sum_{j=1}^{N} \xi_j \varphi_j.$$

Here U is linear linear combination of (a finite number of) basis functions with local support:

$$\{\varphi_j\}_{j=1}^{N}.$$

Some notation from functional analysis

- Scalar product for functions v, w:

$$(v, w) = \int_{\Omega} v(x) w(x) \, dx$$

- $L_2(\Omega)$-norm of a function v:

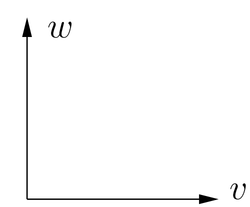
$$\|v\|_{L^2(\Omega)} = \left(\int_{\Omega} v^2 \, dx \right)^{1/2} = \sqrt{(v, v)}$$

Some notation from functional analysis

- Cauchy's inequality:

$$|(v, w)| \leq \|v\| \, \|w\|$$

- v and w are *orthogonal* iff $(v, w) = 0$



Galerkin's method

The finite element method is based on Galerkin's method:

- Let V_h denote a finite dimensional *trial space*.
- Let $\hat{V}_h$ denote a finite dimensional *test space*.
- Find $U \in V_h$ such that the residual $R(U) = A(U) - f$ is orthogonal to $\hat{V}_h$:

$$(R(U), v) = 0 \quad \forall v \in \hat{V}_h.$$

Galerkin's method

For A linear with $V_h = \hat{V}_h = \mathrm{span}\{\varphi_j\}_{j=1}^N$ we have

$$\begin{aligned}
(R(U), v) &= 0, & \forall v \in \hat{V}_h, \\
(A(U) - f, v) &= 0, & \forall v \in \hat{V}_h, \\
(A(\textstyle\sum_{j=1}^N \xi_j \varphi_j), v) &= (f, v), & \forall v \in \hat{V}_h, \\
\textstyle\sum_{j=1}^N \xi_j (A(\varphi_j), \hat{\varphi}_i) &= (f, \hat{\varphi}_i), & i = 1, \ldots, N,
\end{aligned}$$

or

$$A_h \xi = b,$$

where $A_h = (A(\varphi_j), \hat{\varphi}_i)$, $b = (f, \hat{\varphi}_i)$.

Galerkin's method

It is often advisable to rewrite the differential equation $A(u) = f$ from *operator form* to *variational form*:

$$a(u, v) = (f, v) \quad \forall v \in V,$$

where $a(\cdot, \cdot) = (A(\cdot), \cdot)$ is a *bilinear form*, and V is a suitable function space.

Galerkin's method

Starting from the variational formulation, we have

$$\begin{aligned}
a(U, v) = (f, v) &= 0, & \forall v \in \hat{V}_h, \\
a(\textstyle\sum_{j=1}^N \xi_j \varphi_j, v) &= (f, v), & \forall v \in \hat{V}_h, \\
\textstyle\sum_{j=1}^N \xi_j a(\varphi_j, \hat{\varphi}_i) &= (f, \hat{\varphi}_i), & i = 1, \ldots, N,
\end{aligned}$$

or

$$A_h \xi = b,$$

where $A_h = a(\varphi_j, \hat{\varphi}_i)$, $b = (f, \hat{\varphi}_i)$.

FEM for Poisson's equation

Poisson in three different forms

- Equation:

$$-\Delta u = f$$

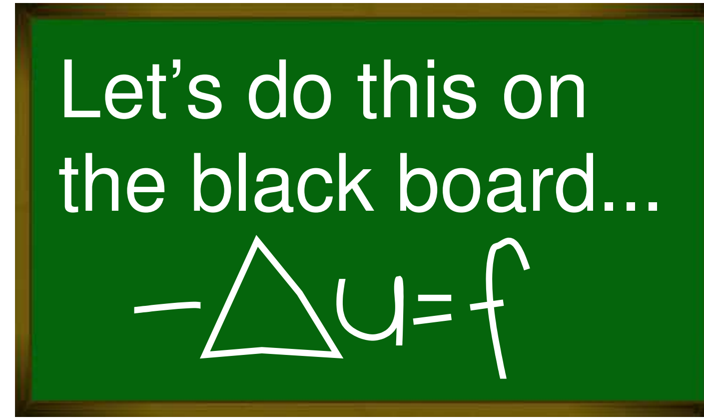
- Variational formulation:

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx \quad \forall v \in V$$

- Linear system:

$$A_h = \int_\Omega \nabla \varphi_j \cdot \nabla \hat{\varphi}_i \, dx, \quad b = \int_\Omega f v \, dx$$

Details



Adaptivity for Poisson

How large is the error?

We expect the error $e = U - u$ to decrease if we increase the dimension N of V_h and $\hat{V}_h$. This can be done in different ways:

- h-adaptivity: decrease the mesh size h
- p-adaptivity: increase the polynomial order p
- hp-adaptivity: a combination of the h and p methods

We will only consider h-adaptivity.

An a posteriori error estimate

Let $\|\cdot\|_E$ denote the *energy-norm* given by $\|v\|_E = \|\nabla v\|$. Then the (piecewise linear) finite element solution $U = U(x)$ satisfies the error estimate

$$\|e\|_E = \|U - u\|_E \leq C\|h(R_1(U) + R_2(U))\|,$$

where $R_1(U) = |f + \Delta U| = |f|$ and

$$R_2(U) = \frac{1}{2} \max_{S \subset \partial K} h_K^{-1} |[\partial_S U]|.$$

Adaptive error control

Find V_h, given by a *triangulation* $\mathcal{T}_h$, such that

$$\|e\|_E \leq \mathrm{TOL},$$

where TOL is a given tolerance for the error.

This is satisfied if

$$C\|h(R_1(U) + R_2(U))\| \leq \mathrm{TOL}.$$

An adaptive algorithm

1. Choose an initial triangulation $\mathcal{T}_h^0$.
2. Compute the solution U on the current triangulation.
3. Compute the residuals R_1, R_2, and the error estimate.
4. If the error estimate is below the tolerance we stop. Otherwise, we refine the elements where $R_1 + R_2$ is large and start again at 2.

FEM for $\dot{u} = f$

$\dot{u} = f$ in three different forms

- Equation:

$$\dot{u}(t) = f(u(t), t)$$

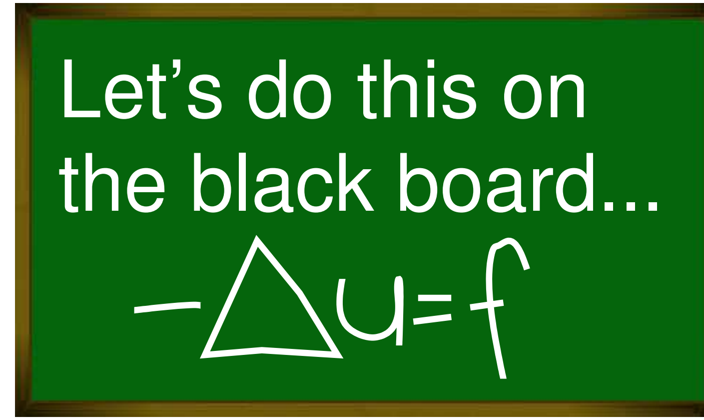
- Variational formulation:

$$\int_{t_{n-1}}^{t_n} (\dot{u}, v)\; dt = \int_{t_{n-1}}^{t_n} (f, v)\; dt \quad \forall v \in V$$

- Step method:

$$U(t_n) = U(t_{n-1}) + \int_{t_{n-1}}^{t_n} f(U(t), t)\; dt$$

Details



Adaptivity for $\dot{u} = f$

An a posteriori error estimate

We expect the error to decrease if we decrease the time step k. The (piecewise linear) finite element solution $U = U(t)$ satisfies the a posteriori error estimate

$$\|e(T)\| = S(T) \max_{[0,T]} \{k(t)\|R(U,t)\|\},$$

where $S(T)$ is a *stability factor* and $R(U,t) = \dot{U}(t) - f(U(t),t)$ is the residual.

An adaptive algorithm

1. Make a preliminary estimate of $S(T)$.
2. Compute the solution U with time steps based on the error estimate.
3. Compute the *dual solution* φ. (See Chapter 9 in CDE.)
4. Compute an error estimate.
5. If the error estimate is below the tolerance we stop. Otherwise start again at 2.