79
Optimal control

We're making the right decisions to bring the solution to an end.
(George W. Bush)

79.1 Introduction

In this chapter we continue with aspects of optimization connected to *optimal control* in the following setting: Consider an IVP of the form: Find the *state* $v : [0, T] \to \mathbb{R}^n$ satisfying the *state equation*

$$\dot{v}(t) + f(v(t), q(t)) = 0 \quad 0 < t \leq T, \quad v(0) = u^0, \tag{79.1}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a given function, u^0 a given initial value, and $q : [0, T] \to \mathbb{R}^m$ is a *control*. We seek to determine an *optimal control* $p : [0, T] \to \mathbb{R}^m$ such that $J(p) \leq J(q)$ for all $q : [0, T] \to \mathbb{R}^m$, where

$$J(q) \equiv \frac{1}{2}\|v - \hat{u}\|^2 + \frac{\alpha}{2}\|q\|^2, \tag{79.2}$$

where v solves (79.1), $\hat{u} : [0, T] \to \mathbb{R}^n$ is a given function, and

$$\|w\|^2 = \int_0^T |w(t)|^2 \, dt$$

with $|\cdot|$ denoting the Euclidean norm, and α is a postive constant. We thus seek to choose the control q so that the corresponding state v is as close as possible to a given state $\hat{u}$ in the $\|\cdot\|$-norm and we also add a cost of the control cost measured by the factor $\alpha > 0$.

We reformulate this problem as the following *saddle point problem*:

$$\min_{v,q} \max_{\mu} L(v, q, \mu) \tag{79.3}$$

with the *Lagrangian* L defined by

$$L(v, q, \mu) = \frac{1}{2}\|v - \hat{u}\|^2 + \frac{\alpha}{2}\|q\|^2 + (\dot{v} + f(v, q), \mu) \tag{79.4}$$

with $(\cdot, \cdot)$ the scalar procuct corresponding to the norm $\|\cdot\|$, and (v, q, μ) varying freely (with $v(0) = u^0$ and $\mu(T) = 0$).

The condition for stationarity of $L(v, q, \mu)$ at (u, p, λ) is $L'(u, p, \lambda) = 0$, where L' is the Jacobian of $L : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$, or in component form:

$$(\dot{u} + f(u, p), \mu) = 0 \quad \forall \mu, \tag{79.5}$$

$$(u - \hat{u}, v) + (\dot{v} + f'_v(u, p)v, \lambda) = 0 \quad \forall v, \tag{79.6}$$

$$(f'_q(u, p)q, \lambda) + \alpha(p, q) = 0 \quad \forall q, \tag{79.7}$$

where $f'_v(v, q)$ and $f'_q(v, q)$ denote the Jacobians of $f(v, q)$ with respect to v and q at (v, q), respectively, and we asume that $v(0) = 0$ and $\mu(T) = 0$. We can restate these equations in (u, p, λ) pointwise in time as follows:

$$\dot{u} + f(u, p) = 0 \quad \text{on } [0, T], \quad u(0) = u^0, \tag{79.8}$$

$$-\dot{\lambda} + f'_v(u, p)^\top \lambda = \hat{u} - u \quad \text{on } [0, T], \quad \lambda(T) = 0. \tag{79.9}$$

$$f'_q(u, p)^\top \lambda + \alpha p = 0 \quad \text{on } [0, T], \tag{79.10}$$

where $\top$ denotes transpose. Here (79.8) is the state equation, (79.9) is the *costate equation*, and (79.14) is the *feed back control* coupling the *optimal control* p to the *costate* λ.

To solve the stationarity equations we may consider the following *gradient method* in the control p:

$$p^{n+1} = p^n - \kappa(\alpha p^n + f'_q(u^n, p^n)^\top \lambda^n) \tag{79.11}$$

where u^n and λ^n solve the state and costate equations $\dot{u}^n + f(u^n, p^n) = 0$ and $-\dot{\lambda}^n + f'_v(u^n, p^n)^\top \lambda^n = \hat{u} - u^n$, respectively, and $\kappa > 0$ is a *step length*.

EXAMPLE 79.1. If $f(v, p) = Av - Bq$ with A a $n \times n$ and B a $n \times m$ matrix, then the stationarity equations take the form:

$$\dot{u} + Au = Bp \quad \text{on } [0, T], \quad u(0) = u^0, \tag{79.12}$$

$$-\dot{\lambda} + A^\top \lambda = \hat{u} - u \quad \text{on } [0, T], \quad \lambda(T) = 0. \tag{79.13}$$

$$\alpha p = B^\top \lambda \quad \text{on } [0, T]. \tag{79.14}$$

79.2 The connection between $\frac{dJ}{dp}$ and $\frac{\partial L}{\partial p}$

We shall now prove that

$$J'(p) = \frac{dJ}{dp}(p) = \frac{\partial L}{\partial p}(u, p, \lambda), \tag{79.15}$$

where the state $u = u(p)$ satisfies the state equation (79.8) with control p, and the costate λ satisfies the costate equation (79.9). We can thus express the gradient $J'(p) = \frac{dJ}{dp}(p)$ of the cost function $J(p)$ in terms of the corresponding state $u = u(p)$ and costate λ, while direct computation of $J'(p)$ requires computation of the derivative $u'(p)$ of the state $u(p)$ with respect to the control p: By the Chain rule we have

$$J'(p)q = \frac{\partial}{\partial \epsilon} J(p + \epsilon q)|_{\epsilon=0} = (u(p) - \hat{u}, u'(p)q) + \alpha(p, q),$$

where we thus want to eliminate $u'(p)$. To do so we differentiate the state equation in the form (assuming for simplicity that $u^0 = 0$),

$$0 = (u, -\dot{\mu}) + (f(u, p), \mu) \quad \forall \mu \text{ with } \mu(T) = 0,$$

with respect to p, to get $\forall \mu$ with $\mu(T) = 0$,

$$0 = \frac{d}{d\epsilon}\big((u(p + \epsilon q), -\dot{\mu}) + (f(u(p + \epsilon q), p + \epsilon q, \mu))\big)|_{\epsilon=0},$$

that is,

$$0 = (u'(p)q, -\dot{\mu}) + (f'_u(u(p), p)u'(p)q + f'_p(u(p), p)q, \mu)$$

or

$$(u'(p)q, -\dot{\mu}) + (u'(p)q, f'_u(u(p), p)^\top \mu) = -(q, f'_p(u(p), p)^\top \mu).$$

Choosing now $\mu = \lambda$ and using that by the costate equation,

$$(u'(p)q, -\dot{\lambda}) + (u'(p)q, f'_u(u(p), p)^\top \lambda) = -(u(p) - \hat{u}, u'(p)q)$$

we can now express $J'(p)$ in the form

$$J'(p)q = (q, f'_p(u(p), p)^\top \lambda) + \alpha(p, q),$$

or

$$J'(p) = f'_p(u(p), p)^\top \lambda + \alpha p = \frac{\partial L}{\partial p}(u, p, \lambda)$$

as we set out to demonstrate.

Through the introduction of the costate λ we are thus able to express the gradient of the cost $J(p)$ with respect to the control p, and we may then apply a gradient method to search for the minimum of $J(p)$.

EXAMPLE 79.2. We consider the problem of balancing an inverted pendelum on a fingertip, when the mass is subject to a perturbation force. Assuming small displacements around the vertical position, the state equation takes the form $\dot{u}_2(t) - u_1(t) = f(t)$ and $\dot{u}_1(t) - u_2(t) = p(t)$ for $0 < t \leq T$, $u_1(0) = u_2(0) = 0$, where $f(t)$ is the perturbation and $p(t)$ the control. The optimal control problem takes the form: Find $p : [0, T] \to \mathbb{R}$ which minimizes the cost

$$J(p) = \frac{1}{2} \int_0^T (u_1^2(t) + u_2^2(t))\, dt + \frac{\alpha}{2} \int_0^T p^2(t)\, dt,$$

where (u_1, u_2) solves the state equation with control p.