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Inverse problems

I never guess. It is a capital mistake to theorize before one has data. Insensibly one begins to twist facts to suit theories, instead of theories to suit facts.

When you have eliminated the impossible, whatever remains, however improbable, must be the truth. (Sherlock Holmes in the *The Sign of Four*, 1888)

78.1 Introduction

We have above in our study of Poisson's equation studied "forward" problems of the form: Given the function $f : \Omega \to \mathbb{R}$, find a function $u : \Omega \to \mathbb{R}$ such that

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ \partial_n u + \kappa u = 0 & \text{on } \Gamma, \end{cases} \tag{78.1}$$

where $\kappa \geq 0$ is a given constant. A corresponding "inverse" problem would be to assume knowledge of $u(x)$ and seek the corresponding function $f(x)$ so that (78.1) is satisfied! If we know $u(x)$ in the entire region Ω, this is a problem of differentiation: we just compute $\Delta u(x)$ from $u(x)$. We then have $-\Delta u = f$! We have studied this problem in Chapter *The derivative*, and we recall that this problem is a bit delicate and that we have to balance the step length h in a difference approximation of Δ to the precision in the given data $u(x)$.

Suppose now that we know $u(x)$ only on the boundary Γ. Can we then determine $f(x)$ in Ω? This type of problem connects to a wealth of important applications of the following form: Suppose we can measure something on the *boundary* of an object. Can we then say something about what is *inside* the object? For example, suppose the object is a human body, and that we can measure something on the boundary (or outside) the body. Can we than get some information on what is inside the body? Or, suppose we can accumulate data on the surface of the Earth, can we then say something about what is in the interior of the Earth, such as the presence of layers of oil. These are all examples of *inverse problems*.

The nature of an inverse problem is to be "ill-posed" in the sense that solutions may be non-unique and/or that small changes in the data may cause large changes in the solution. To single out a unique solution which is not too sensitive to little errors in data, we may have to "regularize" the inverse problem e.g. by smoothing of the data and/or restricting the size of the solution. Differentiation is such an ill-posed problem where we may need to "smooth" or regularize a given function before attempting to compute its derivative.

A typical forward problem is "well-posed" in the sense that small changes in data cause small changes in the solution. A basic example of a well-posed problem is integration corresponding to solving a differential equation. The corresponding inverse problem is differentiation which is ill-posed as we just noted. Solving a differential equation does not always correspond to a well-posed problem: in Chapter *Lorenz and the Essence of Chaos* we met a simple differential equation with solutions being highly sensitive to changes in data.

EXAMPLE 78.1. An *electrocardiogram* ECG produces a curve reflecting the electrical activity of the heart from measurements of electric potentials on the chest, and the curve gives a specialist information on abnormal activities of the heart such as abnormal heart rythm (arrythmias). Similarly, an *electroencephalogram* EEG gives information on the electrical activity of the brain from measurements of electric potentials on the scalp. These techniques are however too imprecise for many diagnoses, and more recently techniques of *electrocardiographic imaging* have been developed, which build on solving inverse problems for Poisson-like equations. The geometry of the individual patient is then obtained from computer tomography, and a picture of the electrical activity inside the body is obtained from measurements of electric potentials on the boundary (e.g. the chest or scalp) by solving an inverse problem for a Poisson-like equation (using the finite element method). Electrocardiographic imaging may give more accurate information on e.g. abnormal cardiac or brain activity than ECG and EEG, and is now a part of practice in advanced neurological and radiological departments. Further development of in particular the computational process (adaptivity, geometric modeling) is needed to increase the accuracy.

EXAMPLE 78.2. Another inverse problem of importance to mankind occurs in inverse seismic prospectation: Explosions on the surface of the Earth are set off and the reflections of the induced waves in the mantle of the Earth are recorded on the surface, and from this information one tries to determine subsurface structures such as layers of oil-bearing rock. To solve this reconstruction problem one uses computational methods based on solving the wave equation involving a wave speed coefficient characteristic of different materials, and through optimization one tries to find the local wave speed coefficient which gives best least squares fit to measured data on the surface of the Earth, and which then gives information on the unknown subsurface layering.

78.2 An inverse source problem for one-dimensional convection

We start considering the simplest boundary value problem:

$$u'(x) = f(x), \text{ for } x \in (0,1],\ u(0) = 0, \tag{78.2}$$

modeling convection with $u : [0,1] \to \mathbb{R}$ representing a concentration and $f : [0,1] \to \mathbb{R}$ a source. We seek to determine or *reconstruct* the function $f : [0,1] \to \mathbb{R}$ from the boundary value *observation* $u(1)$ of the corresponding solution $u(x)$ of (78.2). It is clear that we cannot hope to determine $f(x)$ for all $x \in (0,1)$ from this observation alone. This is because there are many functions $f(x)$ such that the corresponding function $u(x)$ satisfies (78.2) and $u(1) = 0$. To see this it is sufficient to choose a non-zero function $u(x)$ on $[0,1]$ satisfying $u(0) = u(1) = 0$ and define $f(x) = u'(x)$. Evidently, the reconstruction is undetermined up to such functions.

The indeterminancy of the reconstruction $f(x)$ reflects the *ill-posed* nature of the inverse problem; even if the measurements of the boundary value $u(1)$ is very precise, the corresponding source $f(x)$ is not well defined. We thus need some *extra condition* to single out a (hopefully) unique source $f(x)$. We may do this in many ways and depending on the extra condition imposed, we may get different reconstructions $f(x)$. We now indicate one possibility, where we reconstruct under the extra condition that $f(x)$ is *as small as possible* in a least squares sense, which is a common technique of regularization. We then reformulate the inverse problem as the following *least squares optimization problem*: Find the function $f : [0,1] \to \mathbb{R}$ which minimizes the total "cost"

$$J(f) = (u(1) - \bar{u}(1))^2 + \mu \int_0^1 f(x)^2\, dx,$$

where $u(x)$ solves (78.2), $\bar{u}(1)$ is the observed boundary value at $x = 1$, and $\mu > 0$ is a constant. We may view this as a constrol problem where the objective is to find the *control* $f : [0,1] \to \mathbb{R}$ which mimimizes the total cost $J(f)$, where the μ-term measures the cost of the control f and the first term the cost of a boundary value misfit $u(1) - \bar{u}(1)$.

We can phrase this problem as finding the function $u(x)$ with $u(0) = 0$ which minimizes

$$(u(1) - \bar{u}(1))^2 + \mu \int_0^1 (u')^2 \, dx.$$

Recalling Chapter *FEM for two-point boundary value problems*, we understand that the solution $u(x)$ satisfies $u''(x) = 0$ in $(0,1)$, $u(0) = 0$ and $u(1) - \bar{u}(1) + \mu u'(1) = 0$. We conclude that $u(x) = \frac{1}{1+\mu}\bar{u}(1)x$, and thus the reconstructed source $f(x) = u'(x)$ takes on a constant value and is given by

$$f(x) = \frac{1}{1+\mu}\,\bar{u}(1) \quad \text{for } x \in [0,1].$$

Evidently, we are led to choose the regularization parameter μ small; the smaller μ is the more accurately we will fit the boundary value observation $\bar{u}(1)$. Since the reconstructed function $f(x)$ is constant, we have effectively only one constant to determine and we may expect to be able to determine this single value from the single observation $\bar{u}(1)$.

We can also rephrase the optimization problem as follows introducing the integral operator B defined on functions on $[0,1]$ by $Bf(x) = \int_0^x f(y)\,dy$ for $x \in [0,1]$: Find the function $f : [0,1] \to \mathbb{R}$ which minimizes

$$J(f) = (Bf(1) - \bar{u}(1))^2 + \mu \int_0^1 f(x)^2 \, dx.$$

The optimality condition obtained by setting $\frac{d}{d\epsilon}J(f + \epsilon g) = 0$ for $\epsilon = 0$, where $g : [0,1] \to \mathbb{R}$ is an arbitrary function, takes the form:

$$(Bf(1) - \bar{u}(1))Bg(1) + \mu \int_0^1 f(x)g(x)\,dx = 0 \qquad (78.3)$$

for all functions $g : [0,1] \to \mathbb{R}$. We shall now rewrite this condition by introducing the *adjoint operator* $B^\top$ defined on functions $w(x)$ as follows: for a given $w = w(x)$ we let $B^\top w$ be the function on $[0,1]$ satisfying $(B^\top w)'(x) = 0$ for $x \in (0,1)$ and $B^\top w(1) = w(1)$, that is, $B^\top w$ is the constant function on $[0,1]$ taking the value $w(1)$ for all x. We can then rewrite (78.3) in the form

$$\int_0^1 B^\top(Bf(x) - \bar{u}(1))(x)g(x)\,dx + \mu \int_0^1 f(x)g(x)\,dx = 0 \qquad (78.4)$$

for all functions $g:[0,1]\to\mathbb{R}$, because by partial integration

$$\int_0^1 B^\top(Bf(x)-\bar{u}(1))(x)\,\underbrace{(Bg(x))'}_{=g(x)}\,dx=(Bf(1)-\bar{u}(1))Bg(1),$$

where as indicated $(Bg(x))'=g(x)$ and $B^\top w(1)=w(1)$. We conclude that

$$\int_0^1(B^\top Bf+\mu f)g\,dx=\int_0^1 B^\top\bar{u}(1)g\,dx$$

for all functions $g:[0,1]\to\mathbb{R}$, and therefore

$$B^\top Bf+\mu f=B^\top\bar{u}(1)\quad\text{on }(0,1),\tag{78.5}$$

or with I the identity operator:

$$(B^\top B+\mu I)f=B^\top\bar{u}(1).\tag{78.6}$$

We conclude that $f(x)$ is constant on $[0,1]$ and takes the value

$$f(x)=\frac{1}{1+\mu}\,\bar{u}(1)\quad\text{for }x\in[0,1],$$

which is the same result as already derived. We note the form (78.6) of the optimality condition (78.6) with the operator $B^\top B+\mu I$ appearing. We shall meet the same equation below with different solution operators B and adjoints $B^\top$.

78.3 An inverse source problem for one-dimensional diffusion

We continue with the boundary value problem

$$-u''=f\quad\text{in }(0,1),\quad u'(0)=0,\quad u'(1)+u(1)=0,\tag{78.7}$$

where we seek to determine the source $f(x)$ in $(0,1)$ by observing the boundary values $u(0)$ and $u(1)$ of the corresponding solution $u(x)$ of (78.7). Again it is clear that we cannot hope to determine $f(x)$ for all $x\in(0,1)$ from these two observations alone, because there are many functions $f(x)$ such that the corresponding function $u(x)$ satisfies (78.7) and $u(0)=u(1)=0$. To see this it is sufficient to choose a non-zero function $u(x)$ on $[0,1]$ satisfying $u(0)=u'(0)=u(1)=u'(1)=0$ and set $f(x)=-u''(x)$.

As above we seek to reconstruct $f(x)$ under the extra condition that $f(x)$ is as small as possible and we therefore reformulate the inverse problem as the following least squares optimization problem: Find $f(x)$ in $(0,1)$ such that

$$J(f) = (u(0) - \bar{u}(0))^2 + (u(1) - \bar{u}(1))^2 + \mu \int_0^1 f^2(x)\, dx$$

is as small as possible, where $\mu > 0$ is a positive constant acting as a regularization, $\bar{u}(0)$ and $\bar{u}(1)$ are the boundary observations, and of course $u(x)$ solves (78.7). We thus seek $f(x)$ so that in a least squares sense we fit the boundary observations as well as the smallness of $f(x)$ as well as possible.

To state the optimality equations, we introduce the solution operator B corresponding to (78.7), that is, for a given function $f : [0,1] \to \mathbb{R}$ we let $Bf(x)$ be the function on $[0,1]$ satisfying

$$\int_0^1 (Bf)'v'\, dx + Bf(1)v(1) = \int_0^1 fv\, dx, \tag{78.8}$$

for all functions $v(x)$ on $[0,1]$. This follows from Chapter *FEM for two-point boundary value problems*. Setting $\frac{d}{d\epsilon} J(f + \epsilon g) = 0$ for $\epsilon = 0$, we obtain the optimality condition in the form

$$(Bf(0) - \bar{u}(0), Bg(0)) + (Bf(1) - \bar{u}(1), Bg(1)) + \mu \int_0^1 f(x)g(x)\, dx = 0 \tag{78.9}$$

for all functions $g(x)$ on $[0,1]$. Next we introduce the adjoint operator $B^\top$ defined as follows: given the values $w(0)$ and $w(1)$, we let $B^\top w$ be the function on $[0,1]$ which satisfies

$$\int_0^1 (B^\top w)'v'\, dx + B^\top w(1)v(1) = w(0)v(0) + w(1)v(1) \tag{78.10}$$

for all $v(x)$. We see that $(B^\top w)'' = 0$ and $-(B^\top w)'(0) = w(0)$, $B^\top w(1) + (B^\top w)'(1) = w(1)$. In other words, $B^\top w$ is a linear function determined by the two boundary conditions. In particular, if $w(0) = 0$, then $B^\top w = w(1)$ is a constant. Now, setting $v = Bg$ in (78.10), we get

$$\begin{aligned} w(0)Bg(0) + w(1)Bg(1) &= \int_0^1 (B^\top w)'(Bg)'\, dx + B^\top w(1)Bg(1) \\ &= \int_0^1 B^\top wg\, dx, \end{aligned}$$

where we used (78.8) with f replaced by g and v replaced by $B^\top w$, and thus we can write the optimality condition (78.9) in the same form as above:

$$(B^\top B + \mu I)f = B^\top \bar{u}. \tag{78.11}$$

From this equation we can uniquely solve for the function $f(x)$, which will be a linear function defined by two constants, because $f = \frac{1}{\mu} B^\top (Bf - \bar{u})$.

For example if $\bar{u}(0) = 0$ and $\bar{u}(1) = 1$, then we get choosing μ small, $f(x) \approx 10\bar{u}(1)x - 8\bar{u}(1)$ with corresponding solution $u(x) \approx -3x^3 + 4x^2$.

We now comment on the nature of the optimality equation (78.11). The operator B maps a space of sources, say F, into a space of observations, say O, and the adjoint operator $B^\top$ maps O into F. We may think of the dimension of F as large, and that of O as smaller. For the discussion we may assume that the dimension of F is n and the dimension of O is m and thus B corresponds to an $m \times n$ matrix and $B^\top$ to an $n \times m$ matrix with $m << n$. This will be the setting with computational approximations of the solution operators B and $B^\top$. In particular, the columns of B must be severely linearly independent since there are many more columns than rows, and thus the $n \times n$ matrix $B^\top B$ must be singular with many non-zero n-vectors f satisfying $B^\top Bf = 0$. On the other hand, the matrix $B^\top B + \mu I$ with $\mu > 0$ is nonsingular, because if $(B^\top B + \mu I)f = 0$, then scalar multiplication by the n-vector $f^\top$, we obtain $\|Bf\|^2 + \mu\|f\|^2 = 0$ and thus $f = 0$. The non-zero solutions f to $B^\top Bf = 0$ are eigenvectors corresponding to a zero eigenvalue, and by changing to the regularized operator $B^\top B + \mu I$ we shift the spectrum to the interval $[\mu, \infty)$ on the positive real axis.

78.4 An inverse source problem for Poisson's equation

We now pass to an inverse problem for Poisson's equation (78.1) assuming Ω, Γ and κ to be known: Given $u(x) = \hat{U}(x)$ for $x \in \Gamma$, find $f(x)$ for $x \in \Omega$.

We approach this problem directly in discrete form as the following least squares problem: Find $F \in V_h$ which minimizes

$$J(F) = \|U - \hat{U}\|_\Gamma^2 + \mu\|F\|_\Omega^2 \tag{78.12}$$

over V_h, where $U \in V_h$ satisfies

$$(\nabla U, \nabla v)_\Omega + (\kappa U, v)_\Gamma = (F, v)_\Omega \quad \text{for all } v \in V_h, \tag{78.13}$$

and V_h is the space of continuous piecewise linear functions on a given triangulation of Ω of mesh size $h(x)$. As above $\mu \geq 0$ acts as a *regularization parameter* which helps to cope with the ill-posed nature of the problem. Further, $\|\cdot\|_\Omega$ and $(\cdot,\cdot)_\Omega$ denote the $L_2(\Omega)$ norm and scalar product, and similarly, $\|\cdot\|_\Gamma$ and $(\cdot,\cdot)_\Gamma$ denote the $L_2(\Gamma)$ norm and scalar product.

We reformulate (78.12) by introducing the solution operator $B_h : V_h \to W_h$ defined by $B_h F = U_F$ on Γ, where $U_F \in V_h$ solves (78.13), and W_h is the restriction of the space V_h to the boundary Γ, that is a set of piecewise linear functions on Γ. By definition, $U_F \in V_h$ satisfies:

$$(\nabla U_F, \nabla v)_\Omega + (\kappa U_F, v)_\Gamma = (F, v)_\Omega \quad \text{for all } v \in V_h. \tag{78.14}$$

We can now formulate the minimization problem (78.12) as follows: Find $F \in V_h$ which minimizes

$$J(F) = \|B_h F - \hat{U}\|_\Gamma^2 + \mu \|F\|_\Omega^2, \tag{78.15}$$

over V_h. This is a quadratic minimization problem with unique solution $F \in V_h$ characterized by a least squares equation of the form

$$(B_h F, B_h G)_\Gamma + (\mu F, G)_\Omega = (\hat{U}, B_h G)_\Gamma \quad \text{for all } G \in V_h, \tag{78.16}$$

which expresses that $\frac{d}{d\epsilon} J(F + \epsilon G) = 0$ for $\epsilon = 0$ for all $G \in V_h$.

We can express (78.17) as

$$(B_h^\top B_h F, G)_\Omega + (\mu F, G)_\Omega = (B_h^\top \hat{U}, G)_\Omega \quad \text{for all } G \in V_h,$$

that is

$$(B_h^\top B_h + \mu I) F = B_h^\top \hat{U}, \tag{78.17}$$

where $B_h^\top : W_h \to V_h$ is the transpose of B_h defined as follows: Given $w \in W_h$, we let $B_h^\top w \in V_h$ satisfy

$$(\nabla v, \nabla B_h^\top w)_\Omega + (\kappa v,\, B_h^\top w)_\Gamma = (v,\, w)_\Gamma \quad \text{for all } v \in V_h. \tag{78.18}$$

In other words, $B_h^\top w$ is an approximation of the solution z of the Poisson-problem:

$$\begin{cases} -\Delta z = 0 & \text{in } \Omega, \\ \partial_n z + \kappa z = w & \text{on } \Gamma. \end{cases} \tag{78.19}$$

Choosing $v = B_h G$ in (78.18), we get using also (78.13) with $v = B_h^\top w$

$$(B_h G, w)_\Gamma = (\nabla B_h G, \nabla B_h^\top w)_\Omega + (\kappa B_h G,\, B_h^\top w)_\Gamma = (G, B_h^\top w)_\Omega$$

and thus as expected from a transpose

$$(B_h G, w)_\Gamma = (G, B_h^\top w)_\Omega,$$

that is, moving B_h from G onto w brings in the transpose $B_h^\top$.

Solving (78.17) gives an approximation $F(x)$ of the function $f(x)$ we are looking for. We may solve (78.17) by direct matrix inversion if the number of nodes is small, and by some iterative method such as the gradient or the conjugate gradient method for larger problems.

The gradient method takes the form:

$$F^{n+1} = F^n - \alpha((B_h^\top B_h + \mu I) F^n - B_h^\top \hat{U}) = F^n - \alpha(B_h^\top (B_h F^n - \hat{U}) + \mu F^n).$$

In each step we have to compute first $B_h F$ and then $B_h^\top (B_h F - \hat{U})$ corresponding to solving two Poisson problems.

EXAMPLE 78.3. In our first application realized using Matlab we interprete (78.17) as a matrix equation explicitely formed by computing the inverses of the stiffness matrices for the problem (78.14) and the adjoint (78.18), and we then solve this matrix equation to get the nodal values of $F(x)$. One may handle a couple of hundreds of nodes this way. For simplicity, we have considered the case $\Omega = \{(x_1, x_2) : 0 < x_1, x_2 < 1\}$ with $\kappa = 1$ and $f = 0.5 + (x - y)(x + y - 1)$, observed the boundary values of the resulting solution u, and then solved for a reconstruction of the given data f using $\mu = 0.0001$. The result is shown in Fig. 78.1 with reconstruction error ~ 0.032 in f and ~ 0.000176 in the resulting state (boundary values).

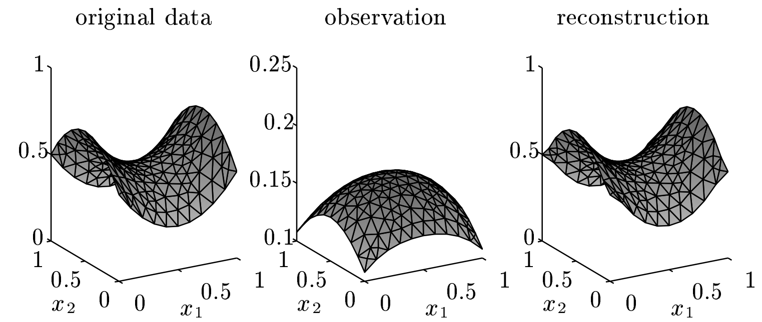

FIGURE 78.1. Original data f (left), resulting observed state u (middle), and the reconstruction of f (right) with $\mu = 0.0001$ and reconstruction error ~ 0.032.

EXAMPLE 78.4. We next take $\kappa = 50$ and observe the resulting state u in Fig. 78.2. The reconstruction using $\mu = 0.0001$ is now rather poor, at least in terms of f with a reconstruction error of order ~ 0.4, while the corresponding state error is of order ~ 0.02. Taking $\mu = 0.00001$ brings the state error down to ~ 0.003, while a reconstruction of f with error ~ 0.04 requires taking $\mu = 0.0000005$.

78.5 An inverse boundary value problem for Laplace's equation

Let Ω be a domain in $\mathbb{R}^2$ with boundary Γ composed of three parts Γ_0, Γ_1 and Γ_2. For a given function f defined on Γ_2, let u_f be the solution to the boundary value problem

$$\begin{cases} -\Delta u_f = 0 & \text{in } \Omega, \\ \quad u_f = 0 & \text{on } \Gamma_0 \cup \Gamma_1, \quad u_f = f \quad \text{on } \Gamma_2, \end{cases} \tag{78.20}$$

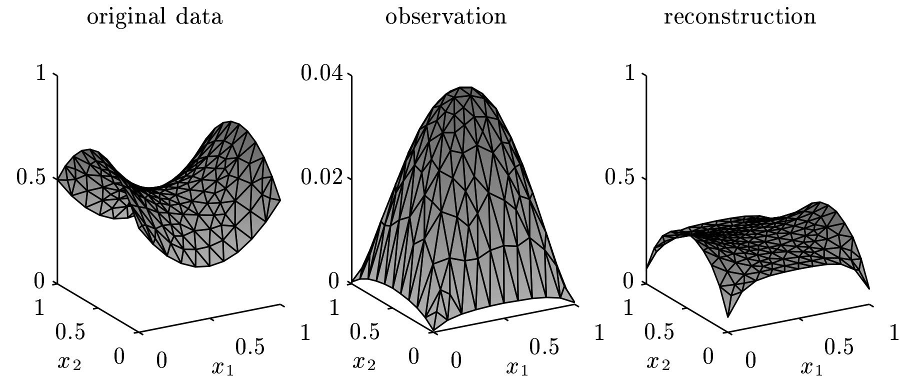

FIGURE 78.2. Original data f (left), resulting observed state u (middle), and the reconstruction of f (right) with $\mu = 0.0001$, now with reconstruction error ~ 0.43.

and define $Bf = \frac{\partial u_f}{\partial n}$ on Γ_1, where n is the unit outward normal to Γ_1. We may think of u_f as a stationary temperature defined in Ω satisfying given boundary conditions on Γ ($= 0$ on $\Gamma_0 \cup \Gamma_1$ and $= f$ on Γ_2) and with Bf representing the heat flux on Γ_1. Suppose now we can measure the heat flux on Γ_1 and that we want to determine the temperature f on Γ_2. We thus have a situation where we have access to the temperature ($= 0$) along $\Gamma_0 \cup \Gamma_1$ and may measure also the heat flux, say $\bar{q}$, along Γ_1, and we want to determine the temperature f on the inaccessible boundary part of the boundary Γ_2. This problem arises in EKG with u being a potential and Γ_1 representing the surface of the chest and Γ_2 that of the heart, and the inverse problem being to reconstruct the potential on the heart from measurements on the chest.

We formulate the reconstruction problem as a least squares optimization problem of the form: Find f on Γ_2 which minimizes

$$J(f) = \|Bf - \bar{q}\|_{\Gamma_1}^2 + \mu \|f\|_{\Gamma_2}^2,$$

where we use the notation of the previous section, and $\mu > 0$. The optimality equation as usual takes the form

$$(B^\top B + \mu I) = B^\top \bar{q},$$

where $B^\top g = \frac{\partial u^g}{\partial n}$ on Γ_2 and u^g solves the problem

$$\begin{cases} -\Delta u^g = 0 & \text{in } \Omega, \\ \quad u^g = g & \text{on } \Gamma_1, \quad u^g = 0 \quad \text{on } \Gamma_0 \cup \Gamma_2. \end{cases} \tag{78.21}$$

This beacause by integrations by parts

$$(g, Bf)_{\Gamma_1} = (\nabla u^g, \nabla u_f)_\Omega = (B^\top g, f)_{\Gamma_2}$$

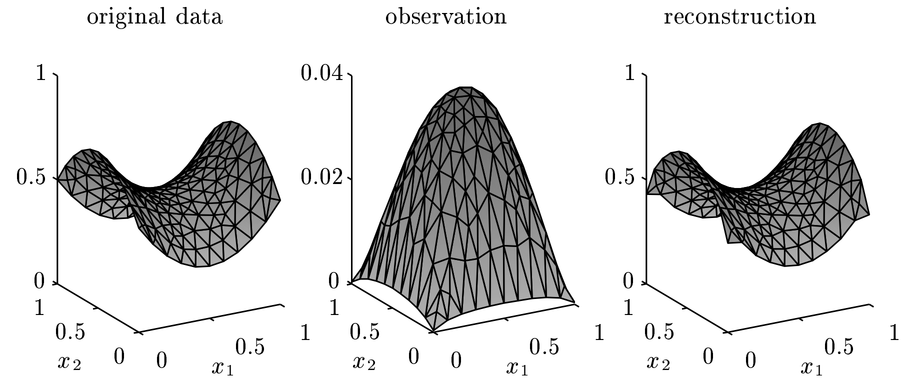

FIGURE 78.3. Original data f (left), resulting observed state u (middle), and the reconstruction of f (right) after 10 steps of the conjugate gradient method with $\mu = 0.000001$, with a reconstruction error ~ 0.0003 in (the boundary values of) u and ~ 0.07 in f.

EXAMPLE 78.5. We consider again the domain $\Omega = \{(x_1, x_2) : 0 < x_1, x_2 < 1\}$ now with $\Gamma_1 = \{(0, x_2) : 0 < x_2 < 1\}$, $\Gamma_2 = \{(1, x_2) : 0 < x_2 < 1\}$ and $\Gamma_0 = \Gamma\backslash\Gamma_1$ with an observed flow $\bar{q}$ along Γ_1 corresponding to $f = 6\,x_2^2\,(1 - x_2)$ along Γ_2. The figure shows the original (Dirichlet) boundary values to the left, the resulting state u and the associated observed flux q along Γ_1 in the middle, and the control/reconstruction f after a few conjugate gradient iterations to the right, with $\mu = 0.001$. The error in the (piecewise constant) reconstruction of the boundary values along Γ_2 is ~ 0.2 and the resulting error in flux through Γ_1 is ~ 0.006.

78.6 The Backward heat equation

Another basic inverse problem is the The *Backward heat equation*: Given the temperature at final time $t = T$, find the temperature at initial time $t = 0$.

We consider this problem in the following setting: let $f(x)$ be an initial temperature and let $u(x, t)$ be the corresponding solution of the heat equation:

$$\begin{cases} \dot{u} - \Delta u = 0 & \text{in } \Omega \times (0, T], \\ \partial_n u + \kappa u = 0 & \text{on } \Gamma \times (0, T], \\ u(x, 0) = f(x) & \text{for } x \in \Omega, \end{cases}$$

where the domain $\Omega \in \mathbb{R}^d$ and the coefficient $\kappa \geq 0$ are given. We consider the following inverse problem: Given the final temperature $u(x, T)$, find

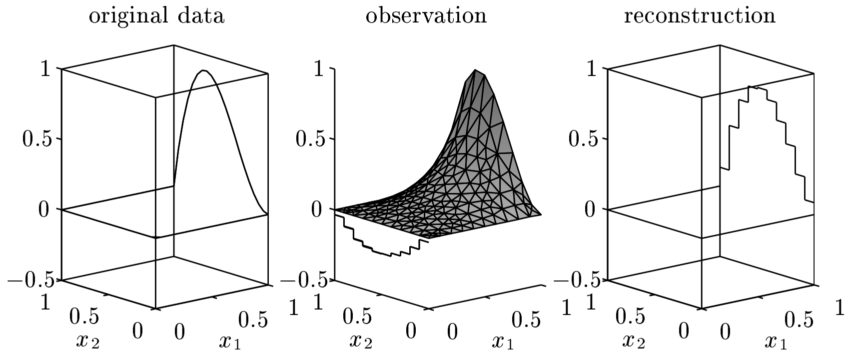

FIGURE 78.4. Original Dirichlet boundary values (left), the corresponding state u with observed flow q along Γ_1 (middle), and the reconstruction of the boundary values along Γ_2 (to the right) using $\mu = 0.001$ and a few (5) conjugate gradient steps.

the initial temperature $u(x,0) = f(x)$. This corresponds to solving the heat equation "backwards".

We consider the following discrete analog of (78.6) with discretization in space: Let V_h be the usual space of continuous piecewise linear functions on a triangulation of Ω with mesh size $h(x)$, and let $F \in V_h$ and let $U(t) \in V_h$ be the solution of the discrete heat equation

$$\begin{cases} (\dot{U}, v)_\Omega + (\nabla U(t), \nabla v)_\Omega = 0 & \text{for } t \in (0,T],\ v \in V_h, \\ (U(0), v)_\Omega = (F, v)_\Omega & \text{for } v \in V_h. \end{cases} \tag{78.22}$$

The discrete inverse problem, corresponding to a discrete "backwards" heat equation, reads: Given the final temperature $U(T) = \hat{U} \in V_h$, find the initial temperature $U(0) = F \in V_h$.

To formulate this problem as a regularized least squares problem, we introduce the solution operator $B_h : V_h \to V_h$ defined as follows: $B_h F = U(T) \in V_h$, where U solves with $U(0) = F \in V_h$. The operator B_h thus takes an initial temperature F to a corresponding final temperature $B_h F$. The regularized least squares problem is now the same as that above, that is, we seek $F \in V_h$ which minimizes

$$J(F) = \|B_h F - \hat{U}\|_\Omega^2 + \mu \|F\|_\Omega^2, \tag{78.23}$$

over V_h. The unique solution $F \in V_h$ to this quadratic minimization problem is characterized by a least squares equation of the form

$$(B_h F, B_h G)_\Omega + (\mu F, G)_\Omega = (\hat{U}, B_h G)_\Omega \quad \text{for all } G \in V_h, \tag{78.24}$$

which we can express as

$$(B_h^\top B_h F, G)_\Omega + (\mu F, G)_\Omega = (B_h^\top \hat{U}, G)_\Omega \quad \text{for all } G \in V_h,$$

that is

$$(B_h^\top B_h + \mu I)F = B_h^\top \hat{U}, \tag{78.25}$$

where $B_h^\top : V_h \to V_h$ is defined as follows: $B_h^\top G = Z(0) \in V_h$, where $Z(t) \in V_h$ solves the discrete heat equation

$$\begin{cases} -(v, \dot{Z})_\Omega + (\nabla v, \nabla Z)_\Omega = 0 & \text{for } t \in (0, T],\ v \in V_h, \\ (v, Z(T))_\Omega = (G, v)_\Omega & \text{for } v \in V_h. \end{cases} \tag{78.26}$$

Note computing $B_h^\top$ corresponds to solving a heat equation: note the minus sign in the term $-(v, \dot{Z})$ and that we solve starting with $t = T$ and ending with $t = 0$. Changing variables introducing a new time variable $s = T - t$ brings this problem into the form of the usual heat equation. Note that (78.26) is a discrete analog of the problem:

$$\begin{cases} -\dot{z} - \Delta z = 0 & \text{in } \Omega \times (0, T], \\ \partial_n z + \kappa z = 0 & \text{on } \Gamma \times [0, T), \\ z(x, T) = g(x) & \text{for } x \in \Omega, \end{cases}$$

with $G \in V_h$ an approximation of $g(x)$ and $Z(t) \in V_h$ an approximation of $z(\cdot, t)$ for $t \in [0, T]$.

To solve the least squares equation by the gradient or conjugate gradient method, we have to compute $B_h F^n$ and $B_h^\top G$ for given vectors F^n and G in V_h, by using some time stepping method such as the dG(0) or cG(1) method.

Example 78.6. We consider the given problem with domain Ω as in the previous examples, $\kappa = 1000$ corresponding to boundary conditions $u \approx 0$, final time $T = 0.1$ and observed state at time T corresponding to initial values $u_0 = 16\, x_1\, (1 - x_1)\, x_2\, (1 - x_2)$ and $u_0 = 2 \min(\, x_1,\, 1 - x_1,\, x_2,\, 1 - x_2)$, respectively. We then seek to reconstruct these initial data from the observations of the resulting solutions at time $T = 0.1$ with $\mu = 0.001$ and a few conjugate gradient iteration using the cG(1) (initiated by two dG(0) steps to filter out high frequency noise) with timesteps $k = 0.0025$. The results are shown in Fig. 78.5 and 78.6, respectively. The reconstruction errror in the first case is ~ 0.057 and in the second case ~ 0.19.

One would think that by decreasing μ it would be possible to better reconstruct the crisp details in the initial data u_0 in the second example. However, the observation we use here is a computed one modelling the fact that observations are imperfect or not considered in full detail in most cases, so that in this case the reconstruction does not get much better by decreasing μ in this case. However, if we decrease T to say 0.02 we can reconstruct also the more detailed structure of u_0 also in the second example with $\mu = 0.00001$ and a resulting reconstruction error ~ 0.068:

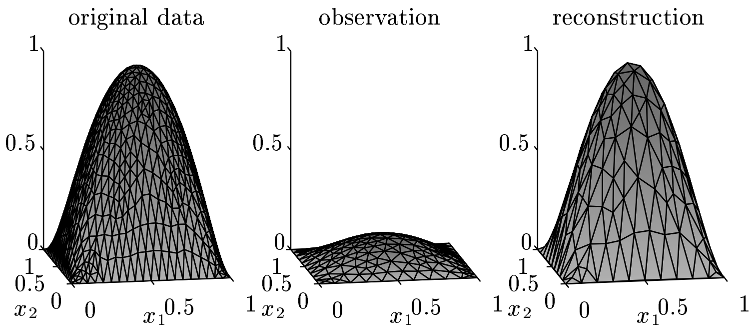

FIGURE 78.5. Original initial data $u_0 = 16\, x_1\, (1 - x_1)\, x_2\, (1 - x_2)$ (left), corresponding observed state/solution at time $T = 0.1$ (middle), and reconstruction of u_0 (right) obtained with $\mu = 0.001$.

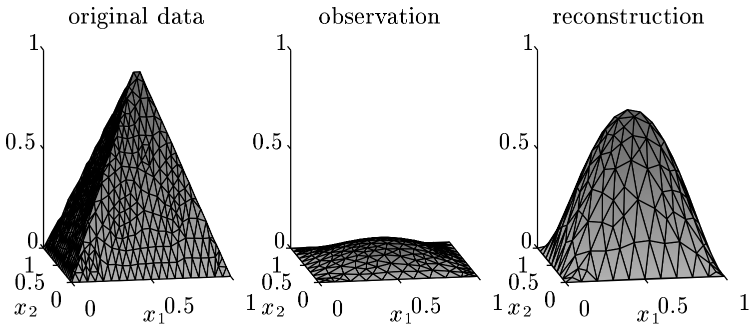

FIGURE 78.6. Original initial data $u_0 = 2\max(\, x_1,\ 1 - x_1,\ x_2,\ 1 - x_2)$ (left), corresponding observed state/solution at time $T = 0.1$ (middle), and reconstruction of u_0 (right) obtained with $\mu = 0.001$.

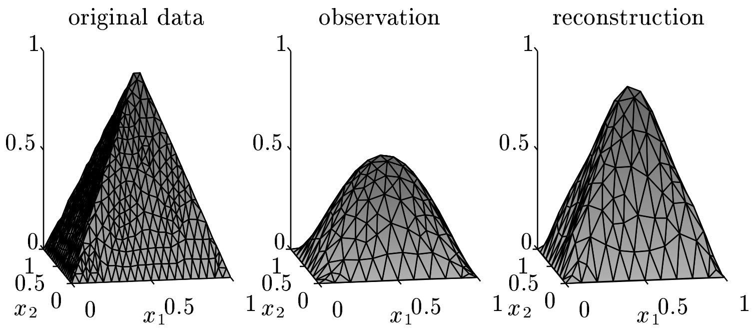

FIGURE 78.7. Original initial data $u_0 = 2\max(\, x_1,\ 1 - x_1,\ x_2,\ 1 - x_2)$ (left), corresponding observed state/solution at time $T = 0.02$ (middle), and reconstruction of u_0 (right) obtained with $\mu = 0.00001$.