Lecture notes in

PDE and FEM

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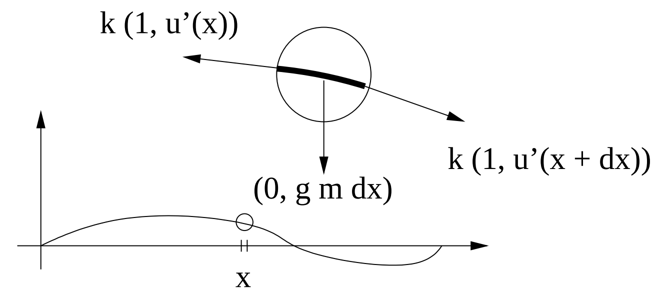
Lecture 1:

PDE models

A vibrating string

- u vertical displacement
- m density, mass per unit length
- g exterior force per unit length
- k string “stiffness”



Balance of (vertical) forces

According to Newton:

$$m\,\ddot{u}\,dx = g\,m\,dx + k\,u'(x+dx,t) - k\,u'(x,t).$$

Dividing by dx and letting $dx \to 0$ gives

$$m\ddot{u} = g + ku''$$

where $\dot{v} = \frac{\partial v}{\partial t}$ & $v' = \frac{\partial v}{\partial x}$.

Special cases

If $g = f - l\,\dot{u}$, with $-l\,\dot{u}$ friction force, we get

$$-ku'' + l\dot{u} + m\ddot{u} = f.$$

Three particular cases:
I. $m = k = 1$, $l = 0$ (no friction)

$$\ddot{u} - u'' = f \qquad \text{The wave equation}$$

II. $l = k = 1$, $m = 0$ (zero weight string)

$$\dot{u} - u'' = f \qquad \text{The heat equation}$$

III. Stationary (time independent) state

$$-u'' = f \qquad \text{Poisson's equation}$$

If $f = 0$ (zero data) the equations are called homogeneous.

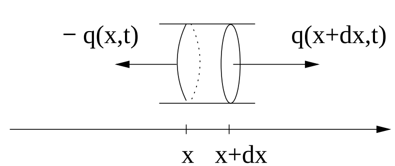
Heat conduction

Heat conduction (one dimensional)

- u temperature
- q heat flux
- f heat sourse
- k heat conductivity

Fouriers law:

$$q = -ku' \qquad (k > 0).$$



The heat equation

Conservation of energy:

$$\dot{u}\,dx = f\,dx - \underbrace{[q(x+dx,t) - q(x,t)]}_{\text{heat outflux}}$$

Dividing with dx and letting $dx \to 0$ gives

$$\dot{u} = f - q' = f - (-k\,u')' = f + (k\,u')'.$$

that is, for $k = 1$,

$$\dot{u} - u'' = f \qquad \text{heat equation}$$

Initial conditions

Must also specify

Initial conditions

- $u(x,0) = u_0(x)$
- $\dot{u}(x,0) = \dot{u}_0(x)$

Note 1: $\dot{u}_0$ is *not* the time derivative of u_0.
Note 2: For the heat equation, the first initial condition is sufficient.

Boundary conditions

Boundary conditions
All around boundary, specify one of

- $u = g$ (given temperature, Dirichlet type)
- $q = -ku' = g$ (given heat flux, Neumann type)
- $q = \sigma(u - \bar{u})$ (flux prop. to heat difference) that is
- $ku' + \sigma u = g$ (Robin or "third" type)

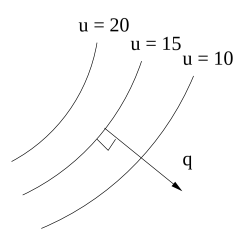
In higher dimensions

In higher space dimension:

$$v' \to \text{grad}\, v = \nabla v = (\frac{\partial v}{\partial x_1}, .., \frac{\partial v}{\partial x_d})$$

For example, Fouriers law takes the form

$$q = -k\nabla u \quad \text{Heat flux}$$



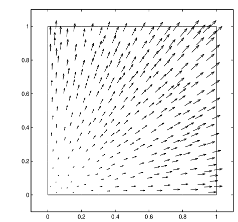
by Gauss theorem

Further

$$q' \to \text{div}\, q = \nabla \cdot q = \frac{\partial q_1}{\partial x_1} + .. + \frac{\partial q_d}{\partial x_d}$$

because by Gauss divergence theorem

$$\int_\gamma q \cdot n = \int_\omega \nabla \cdot q = \int_\omega \text{div}\, q$$



Model cases

In particular

$$(ku')' \to \nabla \cdot k\nabla u = \text{div} \cdot (k\, \text{grad}\, u),$$

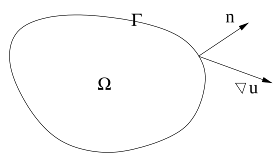
and if $k = 1$,

$$u'' = (u')' \to \nabla \cdot \nabla u = \text{div} \cdot (\text{grad}\, u) = \underbrace{\frac{\partial^2 u}{\partial x_1^2}}_{\frac{\partial}{\partial x_1}\frac{\partial u}{\partial x_1}} +.. + \frac{\partial^2 u}{\partial x_d^2} =: \Delta u$$

Bdry cond in higher dim

Correspondingly, the boundary conditions change as

$$q = g \to \delta_n u := \frac{\partial u}{\partial n} = n \cdot \nabla u = g \qquad \text{Given heat flux}$$



Fluid flow

Incompressible Navier-Stokes

- $u = (u_1, u_2, u_3)$ velocity
- p pressure
- f exterior force
- μ viscosity

Navier-Stokes (system of) equations:

$$\dot{u} + (u \cdot \nabla)\, u = f - \nabla p + \mu \Delta u$$
$$\nabla \cdot u = 0$$

Four equations for the four unknowns u_1, u_2, u_3, p.

Potential flow

If rot $u = 0$ so that $u = -\nabla\phi$ for some potential ϕ (because of the construction

$$\phi(x) = -\int_{\bar{x}}^{x} u\, ds,$$

which is curve independent because rot $u = 0$) then

$$\nabla \cdot u = 0$$

implies that

$$-\Delta u = 0 \quad \text{Poissons, or Laplace equation}$$

Stokes flow

Stokes flow:
If $(u \cdot \nabla)\, u$ is very small, the NS equations reduces to

$$\dot{u} - \mu \Delta u + \nabla p = f$$
$$\nabla \cdot u = 0$$

which is *linear* in u, as opposed to the N.S. eq's with the *non-linear* term $(u \cdot \nabla)\, u$.

Stokes flow, cont.

For the *vorticity* $\omega = \text{rot}\, u = \text{curl}\, u = \nabla \times u$, we then have

$$\dot{\omega} - \mu\Delta\omega = g\,(= \text{rot}\, f) \quad \text{The heat eq again!}$$

From $\nabla \cdot u = 0$ follows that $u = \text{rot}\,\psi$ for some ψ (the *stream function*) and $\nabla \cdot \psi = 0$ if desired. It follows that

$$\omega = \text{rot}\, u = \text{rot}\,\text{rot}\,\psi = \nabla \underbrace{\nabla \cdot \psi}_{=0} - \Delta\psi$$

that is

$$-\Delta\psi = \omega \quad \text{Poissons eq again!}$$

Maxwells equations

- B magnetic field
- E electric field
- ρ charge density
- j current (density)

Maxwell:

$$\begin{aligned} &\nabla \cdot B = 0 \\ &\nabla \cdot E = \rho \\ &\nabla \times E = -\dot{B} \quad \text{Faraday} \\ &\nabla \times B = j + \dot{E} \end{aligned}$$

Stationary Maxwell

Stationary implies $\nabla \times E = 0$, that is, $E = -\nabla\phi$ for some ϕ, that is

$$-\Delta\phi = \rho \quad \text{Poisson again!}$$

More generally:

$$\nabla \cdot B = 0 \to B = \nabla \times A$$

for some vector potential A. Inserted into Faraday, this gives

$$\nabla \times (E + \dot{A}) = 0 \to E + \dot{A} = -\nabla\phi$$

Stationary Maxwell, cont.

Taking the divergence of both sides we obtain

$$\rho + \underbrace{\nabla \cdot \dot{A}}_{-\ddot{\phi}} = -\Delta\phi$$

that is

$$\ddot{\phi} - \Delta\phi = \rho \quad \text{The wave eq again!}$$

The equation for $\nabla \times B$ gives

$$\underbrace{\nabla \times \nabla \times A}_{\nabla \underbrace{\nabla \cdot A}_{-\dot{\phi}} - \Delta A} = j + \underbrace{\dot{E}}_{-\nabla\dot{\phi} - \ddot{A}}$$

that is,

$$\ddot{A} - \Delta A = j \qquad \text{The wave equation again!}$$

Helmholz equations

The ansatz $u(x,t) = v(x)e^{ct}$ in the wave equation

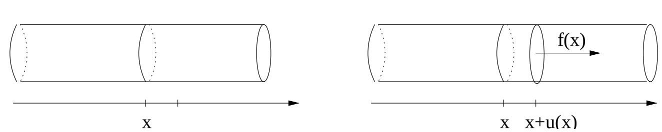
$$\ddot{u} - u'' = f = g(x)\, e^{ct}$$

gives

$$c^2\, v - v'' = g \qquad \text{Helmholtz equation} \qquad (c = i\,\omega)$$

Longitudinel vibration

- $u(x)$ displacement
- $u'(x)$ deformation/strain
- $\sigma = a\,u'(x)$ stress   *Hookes* law
- $f(x)$ load



Balance of forces and Hookes law gives

$$-\sigma' = -(au')' = f.$$

In particular, for $f = 0$ one has $\sigma = \text{const}$.

Elasticity, plane and linear

The displacement and deformation/strain is now given by

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \qquad u' = \nabla u = \begin{bmatrix} u_{1,1} & u_{1,2} \\ u_{2,1} & u_{2,2} \end{bmatrix}$$

and the load and stress by

$$f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \qquad \sigma = \begin{bmatrix} \sigma_{1,1} & \sigma_{1,2} \\ \sigma_{2,1} & \sigma_{2,2} \end{bmatrix}$$

Again, balance of forces gives

$$-\text{div}\,\sigma = f$$

where by Hookes law $\sigma = \sigma(\nabla u)$.

Lecture 2:

Stability or *energy* estimates

The heat equation

Consider the heat equation

(D) $\dot{u} - \Delta u = 0$ in Ω

with boundary conditions $u = 0$ on $\partial\Omega$ and initial conditions $u = u_0$ at time $t = 0$. Multiply (D) by u and integrate over Ω:

$$\int_\Omega u\,\dot{u} - \int_\Omega u\,\Delta u = 0.$$

Here $u\,\dot{u} = \frac{1}{2}\frac{\partial}{\partial t}u^2$ and $-\int_\Omega u\,\Delta u = \int_\Omega \nabla u \cdot \nabla u$ (no boundary term because $u = 0$ on $\partial\Omega$), so that

The heat equation, cont.

$$\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega u^2 + \int_\Omega |\nabla u|^2 = 0 \quad \text{or} \quad \frac{1}{2}\frac{\partial}{\partial t}||u||^2 + ||\nabla u||^2 = 0,$$

where $||\cdot||$ denotes the $L_2(\Omega)$-norm. Time integration from 0 to t gives

$$\frac{1}{2}||u||^2(t) - \frac{1}{2}||u||^2(0) + \int_0^t ||\nabla u||^2 = 0,$$

The heat equation, cont.

that is

$$||u||^2(t) + 2\int_0^t ||\nabla u||^2 = ||u_0||^2.$$

In particular, we have that

$$||u||(t) \leq ||u_0||.$$

Exercise Show that also $||\nabla u||(t) \leq ||\nabla u_0||$. Hint: multiply (D) by $-\Delta u$, rather than by u. Is the inequality valid for $u_0 =$ const. $\neq 0$? Why/why not?

The heat equation, cont.

Exercise Show corresponding estimates for boundary conditions of the form $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$.
Now multiply (D) by $-t\,\Delta u$ and integrate:

$$-t\int_\Omega \Delta u\,\dot{u} + t\int_\Omega (\Delta u)^2 = 0.$$

Here $\int_\Omega \Delta u\,\dot{u} = -\int_\Omega \nabla u \cdot \nabla \dot{u} = -\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega \nabla u \cdot \nabla u$, that is, we may conclude that

$$t\,\frac{1}{2}\frac{\partial}{\partial t}\,||\nabla u||^2 + t\,||\Delta u||^2 = 0,$$

The heat equation, cont.

which we rewrite as

$$\frac{\partial}{\partial t}(t\,||\nabla u||^2) + 2\,t\,||\Delta u||^2 = ||\nabla u||^2.$$

Time integration now yields

$$t\,||\nabla u||^2(t) + 2\int_0^t s\,||\Delta u||^2(s)\,ds = \int_0^t ||\nabla u||^2 \le \frac{1}{2}\,||u_0||^2.$$

In particular,

$$||\nabla u||(t) \le \frac{1}{\sqrt{2t}}\,||u_0||.$$

The heat equation, cont.

Can also be shown that

$$||\Delta u||(t) \le \frac{1}{\sqrt{2t}}\,||u_0||.$$

Exercise Show this! Hint: multiply (D) by $t^2\,\Delta^2 u$ and note that $\Delta u = \dot{u} = 0$ on $\partial\Omega$ when integrating by parts. Alternatively: Differentiate (D) with respect to t and multiply by $t^2\,\dot{u}$.

The heat equation, cont.

From above we have

$$\int_\epsilon^t ||\dot{u}|| \le \frac{1}{\sqrt{2}}\text{log}(\frac{t}{\epsilon})\,||u_0||.$$

By a somewhat refined analysis we find

$$\int_\epsilon^t ||\dot{u}|| = \int_\epsilon^t ||\Delta u|| \le (\int_\epsilon^t s^{-1})^{1/2}(\int_\epsilon^t s\,||\Delta u||^2)^{1/2} \le \sqrt{\text{log}(\frac{t}{\epsilon})}\,||u_0||.$$

For example, for $t = 1$ and $\epsilon = 0.0001$ we have $\int_\epsilon^1 ||\dot{u}|| \le \sqrt{10}\,||u_0|| \approx 2\,||u_0||$.

The heat equation, cont.

Exercise Show that if $\dot{u} - \Delta u = f$, $u_0 = 0$, and u satisfies boundary conditions as above, then $||u||(t) \le \int_0^t ||f||$. Hint: multiply by u and use the fact that $\int_\Omega u\,\dot{u} =$ (Why?) $= ||u||\frac{\partial}{\partial t}||u||$.

Exercise Estimate $||\nabla u||(t)$ by multiplying by $\dot{u}$.

The wave equation

Consider now the wave equation

$$\ddot{u} - \Delta u = 0 \quad \text{in } \Omega,$$

with boundary condition $u = 0$ on $\partial\Omega$, and initial conditions $u = u_0$ and $\dot{u} = \dot{u}_0$ i Ω at time $t = 0$.
Multiply by $\dot{u}$ and integrate over Ω to obtain

$$\int_\Omega \dot{u}\,\ddot{u} - \underbrace{\int_\Omega \dot{u}\,\Delta u}_{\int_\Omega \nabla\dot{u}\cdot\nabla u} = 0,$$

that is,

The wave equation, cont.

$$\frac{\partial}{\partial t}\left(\frac{1}{2}||\dot{u}||^2 + \frac{1}{2}||\nabla u||^2\right) = 0,$$

that is,

$$\frac{1}{2}||\dot{u}||^2 + \frac{1}{2}||\nabla u||^2 = \text{const., indep. of time } t.$$

The total energy $E_{kin} + E_{pot}$ is conserved, where

- $E_{kin} = \frac{1}{2}||\dot{u}||^2$
- $E_{pot} = \frac{1}{2}||\nabla u||^2$

The wave equation, cont.

Exercise Show that also $||\nabla\dot{u}||^2 + ||\Delta u||^2$ is constant, indep. of time t, that is conserved. Hint: multiply by $-\Delta\dot{u}$, or differentiate and then multiply by $\dot{u}$.

Exercise Find a total energy which is conserved for boundary conditions of the form $\frac{\partial u}{\partial n} + u = 0$.

Poissons equation

Consider now

$$-\Delta u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega.$$

Again, multiply by u and integrate $\int_\Omega$:

$$||\nabla u||^2 = \int_\Omega u\,f \leq ||u||\,||f||$$

by Cauchys inequality. But here

$$||u|| \leq C_\Omega ||\nabla u||.$$

Poissons equation, cont.

Proof: Let ϕ satisfy $\Delta\phi = 1$ and $2|\nabla\phi| \le C_\Omega$ in Ω. Then

$$\int_\Omega u^2 = \int_\Omega \Delta\phi\, u^2 = -\int_\Omega \nabla\phi \cdot (2\,u\,\nabla u) \le C_\Omega ||u||\,||\nabla u||,$$

that is

$$||u|| \le C_\Omega ||\nabla u||,$$

and we may conclude that

$$||\nabla u|| \le C_\Omega ||f||.$$

Poissons equation, cont.

Exercise Derive a corresponding estimate for the equation $-\Delta u + u = f$ with boundary conditions $\frac{\partial u}{\partial n} = 0$.

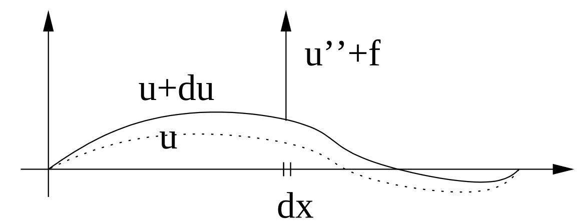
Later on, we shall show that u minimizes the energy

$$\frac{1}{2}||\nabla v||^2 - \int_\Omega f\, v,$$

where $\frac{1}{2}||v||^2$ is "inner" potential energy and $-\int_\Omega f\, v$ is "outer" potential energy.

Poissons equation, cont.

Consider for example a string with displacement $u = u(x)$ as in the figure, and a displacement increment du. Have seen that the vertical force on the string is given by $u'' + f$ per unit length. The displacement increment adds the energy (force $\times$ displacement)



Poissons equation, cont.

$$dE = -\int_{x=0}^{x=1} (u'' + f)\, du\, dx.$$

Integration by parts gives

$$dE = \int_{x=0}^{x=1} (u'\, du' - f\, du)\, dx = d\int_{x=0}^{x=1} (\frac{1}{2}u'^2 - f\, u)\, dx.$$

Exercise Show this by writing $du = \frac{du}{ds}\, ds$, with s a (fictitious) time. During the transition from state $u = 0$ to $u = v$ the total energy added is

$$E = \int_{u=0}^{u=v} dE = \int_{u=0}^{u=v} d\int_{x=0}^{x=1} (d'\, du' - f\, u)\, dx = \int_0^1 \frac{1}{2}v'^2 - f\, v.$$

QED

Lecture 3:

Maximum principles

Max princ for harmonic functions

A solution of *Laplace equation* $\Delta u = 0$ is called *harmonic*.
Principle:

$$\boxed{\text{A (non-constant) harmonic function does not have (interior) local maxima (or minima).}}$$

More precisely: Cannot be a point $\bar{x}$ and an enclosing curve Γ (in R^2, surfact in R^3) such that

$$u(\bar{x}) > \max_\Gamma u,$$

(or

$$u(\bar{x}) < \min_\Gamma u).$$

Max princ for harmonic functions, cont.

Proof: Assume $u(\bar{x}) - \max_\Gamma u = \delta > 0$. For $v(x) := u(x) + \epsilon\,|x - \bar{x}|^2$ and $\epsilon > 0$ sufficiently small, we then have

$$v(\bar{x}) = u(\bar{x}) > \max_\Gamma v.$$

Max princ for harmonic functions, cont.

But then there must be another point $\bar{\bar{x}}$ inside Γ in which v has its maximum, and in such a point

$$\Delta v(\bar{\bar{x}}) \leq 0.$$

On the other hand we know that

$$\Delta v = \underbrace{\Delta u}_{=0} + 2\,d\,\epsilon > 0.$$

This contradiction shows that $u(\bar{x}) > \max_\Gamma u$ cannot hold. QED.

Exercise Show that u such that $-\Delta u + u = 0$ cannot have a *positive* (interior) maximum.

Maximum principle for the heat eq

Principle:

A solution to $\dot{u} - \Delta u = 0$ in $\Omega \times (0, T]$ cannot have a (local) maximum (or min) other than on the "parabolic boundary" $\sqcup := \{(x,t) : x \in \Gamma \text{ or } t = T\}$.

For assume there is $\bar{x} \in \Omega$ and $\bar{t} > 0$ such that

Max principle for the heat eq., cont.

$$u(\bar{x}, \bar{t}) > \max_{\sqcup} u.$$

For v as above and $\epsilon > 0$ sufficiently small, and some point $\bar{\bar{x}}$ and $\bar{\bar{t}}$ we would then have a local max for v and consequently

$$0 \le \underbrace{\dot{v}(\bar{\bar{x}}, \bar{\bar{t}})}_{\ge^* 0} \underbrace{-\Delta v(\bar{\bar{x}}, \bar{\bar{t}})}_{\ge 0}$$
$$= \underbrace{\dot{u}(\bar{\bar{x}}, \bar{\bar{t}}) - \Delta u(\bar{\bar{x}}, \bar{\bar{t}})}_{=0} - 2\, d\, \epsilon < 0,$$

where * is an equality exept possibly for $\bar{\bar{t}} = T$. Again this is a contradiction which shows the claim is right.

Max principle for the heat eq., cont.

Exercise Investigate if the maximum principle can be extended to

$$\dot{u} - \nabla \cdot k \nabla u + \beta \cdot \nabla u = 0, \quad k(x,t) > 0.$$

Fundamental solutions

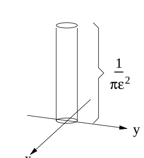
We first look for a solution F to the Poisson equation

$$-\Delta F = \delta, \quad \text{in } R^2,$$

where $\delta = \lim_{\epsilon \to 0} \delta_\epsilon$,

$$\delta_\epsilon(x) = \begin{cases} \frac{1}{\pi \epsilon^2}, & \text{for } 0 \le r \le \epsilon, \\ 0, & \text{for } r > \epsilon, \end{cases}$$

where $r = \sqrt{x_1^2 + x_2^2}$. Note that $\int_{R^2} \delta_\epsilon = 1$:



Fundamental solutions

Exercise Find the corresponding δ_ϵ in R^3.

For $x_1 = r\cos\theta$, $x_2 = r\sin\theta$, we have from the *chain rule*:

$$\Delta F = F''_{x_1x_1} + F''_{x_2x_2} = \frac{1}{r}(r\,F'_r)'_r + \frac{1}{r^2}F''_{\theta\theta}.$$

Here natural to look for solution F independent of θ, that is, to seek $F = F_\epsilon$ such that

$$-\frac{1}{r}\,(r\,F'_r)'_r = \delta_\epsilon,$$

that is

Fundamental solutions

$$(r\,F'_r)'_r = -r\,\delta_\epsilon = \begin{cases} -\frac{r}{\pi\,\epsilon^2}, & \text{for } r \le \epsilon, \\ 0 & \text{for } r > \epsilon. \end{cases}$$

Integration gives

$$r\,F' = \begin{cases} -\frac{r^2}{2\,\pi\,\epsilon^2} + C_1, & \text{for } r \le \epsilon, \\ C_2, & \text{for } r > \epsilon. \end{cases}$$

From putting $r = 0$ we conclude that $C_1 = 0$, and to have continuity for $r = \epsilon$ we conclude that $C_2 = -\frac{1}{2\,\pi}$. Another integration gives

Fundamental solutions

$$F = \begin{cases} \dots \\ \frac{1}{2\,\pi}\log\frac{1}{r}(+D), & \text{for } r > \epsilon, \end{cases}$$

that is, as $\epsilon \to 0$ we obtain

$$F = \frac{1}{2\,\pi}\log\frac{1}{r}(+D).$$

The graph of F looks like a vulcano crater, with $F(r)$ corresponding, e.g., to the temperature field around a point heat sourse at the origin with intensity $\int_{R^2}\delta = 1$:

Fundamental solutions

Remark: There are more solutions, but then also depending on the angle θ.
The corresponding solution in R^3 where

$$\Delta F = \frac{1}{r^2}\,(r^2\,F'_r)'_r + \dots,$$

is given by $F(r) = \frac{1}{4\,\pi}\frac{1}{|x|}$. Again the graph of F looks lika a vulcano.

Exercise Verify the formula for $F(r)$ in R^3.

Superposition

Any function $f = f(x)$ on Ω can be written as the sum/integral of point "loads" as

$$f(x) = \int_\Omega f(y)\,\delta(x-y)\,dy = \int_\Omega f(y)\,\delta(y-x)\,dy.$$

Since the equation is *linear*, a solution to

$$-\Delta u = f \quad \text{in } \Omega$$

can be obtained from the sum/integral of the solutions $f(y)\,F(x-y)\,dy$ corresponding to the individual point loads $f(y)\,\delta(x-y)\,dy$, that is

Superposition

$$u(x) = \int_\Omega f(y)\,F(x-y)\,dy.$$

Note that

$$-\Delta_x F(x-y) = \delta(x-y),$$

which gives

$$\begin{aligned} -\Delta u &= -\Delta_x \textstyle\int_\Omega f(y)\,F(x-y)\,dy \\ &= \textstyle\int_\Omega f(y)\,(-\Delta_x F(x-y))\,dy \\ &= \textstyle\int_\Omega f(y)\,\delta(x-y)\,dy = f. \end{aligned}$$

Superposition

Alternatively,

$$\begin{aligned} u(x) &= \textstyle\int_\Omega u(y)\,\delta(y-x)\,dy = \int_\Omega u(y)\,(-\Delta_y F(y-x))\,dy \\ &= \{\text{Greens formula}\} = \textstyle\int_\Omega \underbrace{(-\Delta u)}_{=f(y)}\,F(y-x)\,dy \\ &= \textstyle\int_\Omega f(y)\,F(x-y)\,dy. \end{aligned}$$

Here $F(x-y) = \frac{1}{2\pi}\log\frac{1}{|x-y|}$ or $F(x-y) = \frac{1}{4\pi}\,\frac{1}{|x-y|}$, depending on the dimension.

Boundary conditions

Consider a domain Ω with boundary Γ:

It follows that also

$$u(x) = \underbrace{\int_\Omega f(y)\,F(x-y)\,dy}_{=:\,v(x)} + \underbrace{\int_\Gamma \phi(y)\,F(x-y)\,dy}_{=:\,w(x)}$$

solves the equation $-\Delta u = f$ in Ω, independent of ϕ, because the "source points" in the w-integral are outside Ω so that w is harmonic (satisfies $\Delta w = 0$) in Ω.

Boundary conditions

The integral w is a called a *single layer potential* and corresponds to the (electric) potential from a “charge” density ϕ on Γ.
To have $u(x) = v(x) + w(x)$ also satisfy given boundary conditions, for example $u = g$ on Γ, we seek ϕ such that

$$\int_\Gamma \phi\, F(x-y)\, dy = g(x) - v(x), \quad \text{for } x \in \Gamma.$$

This is an example of a *Fredholm integral equation of the first kind*

The boundary element method

For simplicity we assume f is zero and consequently $v = 0$.
We make the ansatz $\phi(y) = \sum_{i=1}^m \Phi_i\, \phi_i(y)$, and seek Φ_i such that

$$\int_{K_j} \int_\Gamma \phi(y)\, F(x-y)\, dy\, dx = \int_{K_j} g(x)\, dx, \quad j = 1, .., m,$$

where $\{K_j\}_{j=1}^m$ is an element partition of $\Gamma = K_1 \cup K_2 \cup ... \cup K_m$,

$$\phi_i = \begin{cases} 1, & \text{on } K_i, \\ 0, & \text{on } \Gamma \backslash K_i \end{cases}$$

The boundary element method

This gives a linear system of equations $A\,\Phi = b$ for $\Phi = (\Phi_i)$ with $b_j = \int_{K_j} g(x)\, dx$ and

$$A_{ji} = \int_{K_j} \int_{K_i} F(x-y)\, dy\, dx.$$

An integral equation of the second kind

Fredholm integral equation of the second kind:

$$\phi(x) + \int_\Gamma \phi(y)\, K(x,y)\, dy = h(x),$$

arises for example seeking to satisfy boundary conditions of the for $\partial_n u + \sigma\, u = g$.

Lecture 4:

Fundamental solutions

Fund. sol. f. the heat eq

Claim: For a suitable constant c the function

$$u(x,t) = c\,\frac{1}{t^{d/2}}\,e^{-|x|^2/(4t)}$$

solves the heat equation $\dot{u} - \Delta u = 0$ in R^d, $t > 0$ with initial condition $u(x,0) = \delta(x)$, where δ is Diracs delta function.
Check:

$$\dot{u} = c\,(-d/2)\,\frac{1}{t^{d/2+1}}\,e^{-|x|^2/(4t)} + c\,\frac{1}{t^{d/2}}\,e^{-|x|^2/(4t)}\,\frac{|x|^2}{4t^2},$$

$$\nabla u = c\,\frac{1}{t^{d/2}}\,e^{-|x|^2/(4t)}\,(-\frac{x}{2t}),$$

Fund. sol. f. the heat eq, cont.

$$\Delta u = \nabla\cdot\nabla u = c\,\frac{1}{t^{d/2}}\,e^{-|x|^2/(4t)}\,\underbrace{(-\frac{x}{2t})\cdot(-\frac{x}{2t})}_{|x|^2/(4t^2)} + \frac{1}{t^{d/2}}\,e^{-|x|^2/(4t)}\,(-\frac{d}{2t}).$$

That is

$$\dot{u} = \Delta u \text{ !!}$$

We scetch $u(x,0.01)$, $u(x,1)$ and $u(x,100)$, to display the time evolution of $u(x,t)$. We note, in particular, that

$$u(x,t) \to \begin{cases} 0 & \text{as } t \to 0^+ \text{ for } x \neq 0 \\ \infty & \text{as } t \to 0^+ \text{ for } x = 0. \end{cases}$$

Fund. sol. f. the heat eq, cont.

We next seek to determine c for $d = 2$ such that $u(x,t) \to \delta(x)$ as $t \to 0+$:

$$\int_{R^2} u(x,t)\,dx = c\int_{R^2} \tfrac{1}{t}\,e^{-|x|^2/(4t)}\,dx = 2\pi c\int_0^\infty \tfrac{1}{t}\,e^{-r^2/(4t)}\,r\,dr$$
$$= \left\{\begin{array}{l} s = r^2/(4t) \\ ds = r/(2t)\,dr \end{array}\right\} = 4\pi c\int_0^\infty e^{-s}\,ds = 4\pi c = 1,$$

if $c = 1/(4\pi)$.
Exercise Determine c for $d = 1$ and $d = 3$.

Superposition

The solution to $\dot{u} - \Delta u = 0$, $t > 0$, $u(x,0) = u_0(x)$ is now given by

$$\int_{R^d} u_0(y)\, c\, \frac{1}{t^{d/2}}\, e^{-|x-y|^2/(4t)}\, dy$$

Denote this function (of x) by $E(t)u_0$, that is, $E(t)v$ is a function of x (and t) such that $E'(t)v - \Delta E(t)v = 0$ and $E(0)v = v$, for any v. Using this notation we now get

DuHamels principle

The solution to $\dot{u} - \Delta u = f$ for $t > 0$ and $u(\cdot,0) = 0$ is given by

$$u(x,t) = \int_0^t E(t-s)\, f(\cdot,s)\, ds.$$

To prove this, note that $u(x,0) = 0$. Further, by the properties of $E(t)$ and the differentiation rule

$$\frac{d}{dt}\int_0^t g(t,s)\, ds = g(t,t) + \int_0^t g_t'(t,s)\, ds,$$

DuHamels principle

we find that

$$\dot{u} = E(t-t)\, f(\cdot,t) + \int_0^t \underbrace{E'(t-s)\, f(\cdot,s)}_{=\Delta E(t-s)\, f(\cdot,s)}\, ds$$
$$= f(\cdot,t) + \Delta \int_0^t E(t-s)\, f(\cdot,s)\, ds = f + \Delta u.$$

QED

Superposition

The solution to $\dot{u} - \Delta u = f$ for $t > 0$, $u(x,0) = u_0(x)$ is now given by

$$u(\cdot,t) = E(t)u_0 + \int_0^t E(t-s)\, f(\cdot,s)\, ds,$$

that is

$$u(x,t) = c\,\frac{1}{t^{d/2}} \int_{R^d} e^{-|x-y|^2/(4t)}\, u_0(y)\, dy$$
$$+c\int_0^t \frac{1}{(t-s)^{d/2}} \int_{R^d} e^{-|x-y|^2/(4(t-s))}\, f(y,s)\, dy\, ds.$$

Exercise Compute $u(x,t)$ for $f(x,t) = \delta(x)\,\delta(t)$.

Greens functions

Let $G = G(x,y)$ denote the solution to

$$-\Delta_x G = \delta(x-y) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega.$$

Then, by superposition, the solution to

$$-\Delta u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,$$

is given by

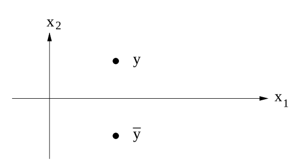
$$u(x) = \int_\Omega G(x,y)\, f(y)\, dy.$$

A Greens function example

Example For $\Omega = \{(x_1,x_2) : x_2 > 0\}$, the *Greens function* $G(x,y)$ is given by

$$G(x,y) = \frac{1}{2\pi}\log(\frac{1}{|x-y|}) - \frac{1}{2\pi}\log(\frac{1}{|x-\bar{y}|}).$$

where $\bar{y}$ is the mirror point of y in the x_1-axis, as in the figure



Exercise Verify this!

The wave equation

For $f = 0$ and one space dimension the wave equation takes the form

$$\ddot{u} - u'' = 0.$$

We show that

$$u(x,y) = g(x-t) + h(x+t)$$

solves the equation, for any functions g and h:

$$u'' = g''+h'', \quad \dot{u} = g' \underbrace{(-1)}_{!} + h', \quad \ddot{u} = g''(-1)(-1)+h'' = g''+h'',$$

where the (-1)-factors are inner derivatives. That is $\ddot{u} = u''$!!.

The wave equation, cont.

Given initial conditions $u(x,0) = u_0(x)$ and $\dot{u}(x,0) = \dot{u}_0(x)$ we can determine g and h: For $\dot{u}_0 = 0$ we have $g' = h'$, that is $h = g + c$ for some constant c, that is $u = g + g + c = 2g + c$. In particular, $u_0 = 2g + c$, that is $g = \frac{1}{2}(u_0 - c)$, and consequently $h = \frac{1}{2}(u_0 + c)$, and we may conclude that

$$u(x,t) = \frac{1}{2}(u_0(x-t)-c) + \frac{1}{2}(u_0(x+t)+c) = \frac{1}{2}u_0(x-t) + \frac{1}{2}u_0(x+t).$$

The wave equation, cont.

Correspondingly, for $u_0 = 0$ we have that $h = -g$, that is $\dot{u}_0 = -g' + h' = 2\,h'$, giving

$$h(z) = \frac{1}{2}\int_0^z \dot{u}_0 + c, \quad g(z) = -\frac{1}{2}\int_0^z \dot{u}_0 - c = \frac{1}{2}\int_z^0 \dot{u}_0 - c,$$

and we may conclude that

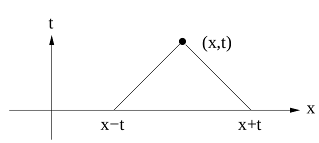
$$u(x,t) = \frac{1}{2}\int_{x-t}^0 -c + \frac{1}{2}\int_0 x + t + c = \frac{1}{2}\int_{x-t}^{x+t} .$$

The wave equation, cont.

We may now again use superposition to conclude that in the general case the solution is given by

$$u(x,t) = \frac{1}{2}u_0(x-t) + \frac{1}{2}u_0(x+t) + \frac{1}{2}\int_{x-t}^{x+t} \dot{u}_0(s)\,ds.$$

Note, in particular, that the solution at (x,t) depends on initial data only within a bounded region $[x-t, x+t]$, that is, we have a finite speed of propagation of information. Compare with solutions of the heat equation where the speed of propagation is infinite!



The wave equation, cont.

Exercise Verify that

$$u(x,t) = \frac{1}{2}\int_0^t \int_{x-t+s}^{x+t-s} f(y,s)\,dy\,ds$$

solves the problem $\dot{u} - u'' = f$. What are the initial values (at $t = 0$) of u and $\dot{u}$? Give a formula for the solution in the general case, with nonzero f, u_0 and $\dot{u}_0$.
Exercise Generalize to the case of equation $\ddot{u} - c^2 u'' = f$.

Boundary value problems

We now seek $u(x,t)$ such that $\ddot{u} - u'' = 0$ for $x > 0$ and $t > 0$, $u(x,0 = u_0(x)$ and $\dot{u}(x,0) = \dot{u}_0(x)$ for $x > 0$, and $u'(0,t) = 0$ for $t > 0$.
To solve this problem we *extend* $u(x,t)$ to an *even* function of x, denoted $\tilde{u}(x,t)$, such that $\tilde{u}(x,t) = u(x,t)$ for $x > 0$ and $\tilde{u}(x,t) = u(-x,t)$ for $x < 0$. By inspection $\tilde{u}$ solves the wave equation for all x, and $t > 0$.
Exercise Verify this!
According to d'Alemberts formula we have

$$\tilde{u}(x,t) = \frac{1}{2}\tilde{u}_0(x-t) + \frac{1}{2}\tilde{u}(x+t) + \frac{1}{2}\int_{x-t}^{x+t} \tilde{v}_0(s)\,ds,$$

where $\tilde{u}_0(s) = \tilde{u}_0(s,0)$, $\tilde{v}_0(s) = \tilde{v}_0(s,0) = \dot{\tilde{u}}(s,0)$

Boundary value problems, cont.

In particular, for $x > 0$ and $x < t$ we obtain

$$u(x,t) = \tilde{u}(x,t) = \tfrac{1}{2}u_0(-(x-t)) + \tfrac{1}{2}u_0(x+t)$$
$$+\tfrac{1}{2}\int_{x-t}^{0} v_0(-s)\,ds + \tfrac{1}{2}\int_0^{x+t} v_0(s)\,ds$$
$$= \tfrac{1}{2}u_0(t-x) + \tfrac{1}{2}u_0(x+t) + \tfrac{1}{2}\int_0^{t-x} v_0(s)\,ds + \tfrac{1}{2}\int_0^{x+t} v_0(s)\,ds.$$

For $x > t$ the solution formula is the same as before. The boundary conditions does not influence the solution in this region.
Exercise Derive the corresponding solution formula for the boundary condition $u(0,t) = 0$.

Boundary value problems, cont.

Exercise Consider the wave equation $\ddot{u} - \Delta u = 0$ in R^3 with radiell symmetry, that is, $u = u(r,t)$ with $r = |x| = \sqrt{x_1^2 + x_2^2 + x_3^2}$. For $u(r,t)$ we then have $\ddot{u} - \frac{1}{r^2}(r^2 u')' = 0$, where $v' = \frac{\partial v}{\partial r}$. With the ansatz $v(r,t) = r\,u(r,t)$ we obtain $\ddot{v} - v'' = 0$. Verify this!

Exercise Find $u(x,t)$ such that $\ddot{u} - \Delta u = 0$ in R^3 for $t > 0$ subject to the initial conditions $u(x,0) = 1$ for $|x| < 1$, $u(x,0) = 0$ for $|x| > 0$, $\dot{u}(x,0) = 0$, corresponding to the pressure wave from a ballon of radius 1 exploding at time $t = 0$.

Examples

Below we give some examples of *single layer potentials*, that is, solutions to $\Delta u = 0$ of the form

$$u(x) = \int_\Gamma \phi(y)\,F(x-y)\,dy,$$

and solutions to $-\Delta u = f$ in Ω of the form

$$u(x) = \int_\Omega f(y)\,F(x-y)\,dy,$$

where

$$F(x-y) = \frac{1}{4\pi}\frac{1}{|x-y|}.$$

Examples

Example 1 With $\Gamma = \{y_1^2 + y_2^2 = r^2,\ y_3 = 0\}$, $y = (r\cos\theta, r\sin\theta, 0)$, $dy = r\,d\theta$, $x = (0,0,R)$, $|x-y|^2 = r^2 + R^2$, $\phi \equiv 1$, we obtain

$$u(x) = \frac{1}{4\pi}\int_0^{2\pi} \frac{1}{\sqrt{r^2+R^2}}\,r\,d\theta = \frac{1}{2}\frac{r}{\sqrt{r^2+R^2}}.$$

Examples

Example 2 With $\Gamma = \{y_1^2 + y_2^2 \le \bar{r}^2,\ y_3 = 0\}$, $y = (r\cos\theta, r\sin\theta, 0)$, $dy = r\,dr\,d\theta$, $x = (0,0,R)$, $|x-y|^2 = r^2 + R^2$, $\phi \equiv 1$, we obtain

$$u(x) = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\bar{r}} \frac{r\,dr\,d\theta}{\sqrt{r^2+R^2}} = \frac{1}{2}\,(\sqrt{\bar{r}^2+R^2} - R).$$

Remark: $u(x) - u(0) = \frac{1}{2}\,(\sqrt{\bar{r}^2+R^2} - R - \bar{r}) \to -\frac{1}{2}R$ as $\bar{r} \to \infty$.

Examples

Example 3 With $\Gamma = \{y_1^2 + y_2^2 + y_3^2 = r^2\}$, $y = (r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi)$, $dy = r^2\sin\phi\,d\phi\,d\theta$, $x = (0,0,R)$, $|x-y|^2 = r^2 + R^2 - 2\,r\,R\cos\phi$, (Check this !!) $\phi \equiv 1$, we obtain

$$\begin{aligned} u(x) &= \tfrac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \tfrac{r^2\sin\phi\,d\theta\,d\phi}{\sqrt{r^2+R^2-2\,r\,R\cos\phi}} \\ &= \tfrac{r}{2\,R}\,[\sqrt{r^2+R^2-2\,r\,R\cos\phi}]_0^{\pi} \\ &= \tfrac{r}{2\,R}\,(r + R - |r-R|) = \begin{cases} r^2/R, & \text{for } R > r, \\ r, & \text{for } R < r \end{cases} \end{aligned}$$

Remark: For $\phi = \frac{1}{4\,\pi\,r^2}$ we obtain

$$u(x) = \begin{cases} \frac{1}{4\,\pi\,R} & \text{for } R > r, \\ \frac{1}{4\,\pi\,r}, & \text{for } R < r. \end{cases}$$

Examples

Example 4 With $\Gamma = \{y_1^2 + y_2^2 + y_3^2 \le \bar{r}^2\}$, $y = (r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi)$, $dy = r^2\sin\phi\,dr\,d\phi\,d\theta$, $|x| = R$, $\phi \equiv 1$, we obtain

$$u(x) = \int_0^{\bar{r}} \frac{r^2}{R}\,dr = \frac{\bar{r}^3}{3\,R}, \quad \text{for } R > \bar{r},$$

$$u(x) = \int_0^{R} \frac{r^2}{R}\,dr + \int_R^{\bar{r}} r\,dr = R^2/3 + \bar{r}^2/2 - R^2/2, \quad \text{for } R < \bar{r}.$$

Remark: For $\phi = 3/(4\,\pi\,\bar{r}^3)$ we get $u(x) = \frac{1}{4\,\pi\,R}$ for $R > \bar{r}$.

Examples

Example 5 With $\Gamma = \{y_1^2 + y_2^2 = r^2,\ -1 \le y_3 \le 1\}$, $y = (r\cos\theta, r\sin\theta, z)$, $dy = r\,d\theta\,dz$, $x = (0,0,R)$, $|x-y|^2 = r^2 + (R-z)^2$, $\phi \equiv 1$, we obtain

$$\begin{aligned} u(x) &= \tfrac{1}{4\pi}\int_{-1}^{1}\int_0^{2\pi} \tfrac{r\,d\theta\,dz}{\sqrt{r^2+(R-z)^2}} = \{u = R - z\} = \tfrac{r}{2}\int_{R-1}^{R+1}\tfrac{du}{r^2+u^2} \\ &= \tfrac{r}{2}(\log|R+1+\sqrt{(R+1)^2+r^2}| - \log|R-1+\sqrt{(R-1)^2+r^2}|). \end{aligned}$$

Examples

Example 6 With $\Gamma = \{y_1 = y_2 = 0, \ -\bar{z} \le y_3 \le \bar{z}\}$, $dy = dz$, $x = (R, 0, 0)$, $\phi \equiv 1$, we obtain

$$u(x) = \frac{1}{4\pi} \int_{\bar{z}}^{\bar{z}} \frac{dz}{z^2 + R^2} = \frac{1}{2\pi} (\log|\bar{z} + \sqrt{\bar{z}^2 + R^2}| - \log R).$$

Remark: $u(x) - u(1) \to -\frac{1}{2\pi} \log R$ as $\bar{z} \to \infty$.

Lecture 5:

Riez and Lax-Milgrams theorems

Equivalent *variational* formulations

Consider the problems: Given Ω and f,

(D) $\quad -\Delta u = f$ in Ω, $\quad u = 0$ on $\partial\Omega$, $\quad u = ?$

and

(M) $\quad F(u) = \min_{v \in V} F(v), \quad u \in V, \quad u = ?$

where

$$F(v) = \frac{1}{2} \int_\Omega |\nabla v|^2 - \int_\Omega v\, f$$

and $V = H_0^1(\Omega) = \{v : \int_\Omega |\nabla v|^2 \le \infty, \, v = 0$ on $\partial\Omega\}$.

Will show that (D)⇔(M).

Equivalent *variational* formulations, cont.

Define for an arbitrary $v \in V$ and $\lambda \in R$

$$g(\lambda) = F(u + \lambda v) = \frac{1}{2} \int_\Omega |\nabla(u + \lambda v)|^2 - \int_\Omega (u + \lambda v)\, f$$

$$= F(u) + \lambda \int_\Omega (\nabla v \cdot \nabla u - v\, f) + \frac{1}{2}\lambda^2 \int_\Omega |\nabla v|^2.$$

Assume now that u solves (M). For

$$g'(\lambda) = \int_\Omega (\nabla v \cdot \nabla u - v\, f) + \lambda \int_\Omega |\nabla v|^2$$

we then have $g'(0) = 0$, that is

Equivalent *variational* formulations, cont.

(V) $$\int_\Omega \nabla v \cdot \nabla u = \int_\Omega v\, f \quad \forall v \in V.$$

But from here follows that

$$\int_\Omega v\,(-\Delta u) = \int_\Omega v\, f, \quad \forall v \in V,$$

which can hold only if $-\Delta u = f$ in Ω, that is, we have showed that (M) $\rightarrow$ (V) $\rightarrow$ (D).

Equivalent *variational* formulations, cont.

On the other hand, clearly $-\Delta u = f$ and $u = 0$ on $\partial\Omega \Rightarrow$ (V), and (V) $\Rightarrow$ (M), because (V) $\Rightarrow$

$$F(v) = F(u+w) = F(u) + \underbrace{\int_\Omega (\nabla w \cdot \nabla u - w\, f)}_{=0} + \underbrace{\frac{1}{2}\int_\Omega |\nabla w|^2}_{\geq 0} \geq F(u),$$

for all $v \in V$ (and $w = v - u \in V$). That is, we have shown that

(D) $\Leftrightarrow$ (V) $\Leftrightarrow$ (M).

Next: An equation does not always have a solution. Does (D), (V) and (M) have a solution?

Existence and uniqueness of solution

Note that $(v,u) := \int_\Omega \nabla v \cdot \nabla u$ is *bilinear*:

$$\begin{aligned} (v_1+v_2, u) &= (v_1,u) + (v_2,u), \quad & (\alpha\, v, u) &= \alpha\,(v,u), \\ (v, u_1+u_2) &= (v,u_1) + (v,u_2), \quad & (v, \beta u) &= \beta(v,u), \end{aligned}$$

symmetric:

$$(v,u) = (u,v),$$

and *positive definite*:

$(v,v) \geq 0,$ and $(v,v) = 0$ only for $v = 0,$

that is $(v,u) = \int_\Omega \nabla v \cdot \nabla u$ is a *scalar product* on V, with corresponding *norm* $||v|| = (v,v)^{1/2}$.

Existence and uniqueness of solution

Further, the right hand side in (V), that is $l(v) = \int_\Omega v\, f$, is *linear* in v:

$$l(v_1+v_2) = l(v_1) + l(v_2), \quad l(\alpha\, v) = \alpha\, l(v),$$

and, if for example $\int_\Omega f^2 < \infty$, *bounded*:

$|l(v)| \leq \gamma\, ||v||,$ with $\gamma = C_\Omega\,(\int_\Omega f^2)^{1/2},$

because

$$\int_\Omega v\, f \leq (\int_\Omega v^2)^{1/2}\,(\int_\Omega f^2)^{1/2} \leq C_\Omega\, ||v||\,(\int_\Omega f^2)^{1/2}.$$

Existence and uniqueness of solution

Back to the question; Is there a $u \in V$ such that

$$(v,u) = l(v), \quad \forall v \in V,$$

or, equivalently, such that

$$F(u) = \min_{v\in V} F(v), \quad F(v) = \frac{1}{2}||v||^2 - l(v).$$

The answer is *yes*, and the argument as follows:

Existence and uniqueness of solution

Take a sequence $u_i \in V$, $i = 1, 2..$ such that

$$F(u_i) \to \sigma := \inf_{v\in V} F(v).$$

Here $\sigma > -\infty$, because

$$F(v) = \frac{1}{2}||v||^2 - l(v) \geq \frac{1}{2}||v||^2 - \gamma||v|| \geq -\frac{1}{2}\gamma^2.$$

Existence and uniqueness of solution

Further

$$\begin{aligned}
&||u_i - u_j||^2 = 2||u_i||^2 + 2||u_j||^2 - ||u_i + u_j||^2 \\
&\underbrace{- 4\,l(u_i) - 4\,l(u_j) + 4\,l(u_i + u_j)}_{=0} \\
&= 4\,f(u_i) + 4\,F(u_j) - 8\,F(\tfrac{u_i+u_j}{2}) \\
&\leq 4\,F(u_i) + 4\,F(u_j) - 8\,\sigma \to 0, \quad \text{as } i,j \to \infty,
\end{aligned}$$

that is, u_i is a *Cauchy sequence* in V, and $V = H_0^1(\Omega)$ is a *Hilbert space*, that is, a *complete* scalar product space, where all Cauchy sequences converge to something.

Existence and uniqueness of solution

That is, there exist $u \in V$ such that $u_i \to u$, and since $F(u_i) \to \sigma$ and F is continuous, we must have that

$$F(u) = \lim_{i\to\infty} F(u_i) = \sigma = \min_{v\in V} F(v)$$

QED.

This result, that every linear and continuous functional $l(v)$ is given by the scalar products (v,u) of some $u \in V$, is called *Riesz theorem*.

More generally, we have *Lax-Milgrams theorem*:

Lax-Milgrams theorem

Assume $l(v)$ is linear and bounded on V and that $a(v,u)$ is bilinear, bounded and positive definite on V, that is

$$|a(v,u)| \le \beta\,||v||\,||u||, \quad \text{for some } \beta \text{ and } \forall v,u \in V,$$

and

$$a(v,v) \ge \alpha\,||v||^2, \quad \text{for some } \alpha > 0 \text{ and } \forall v \in V.$$

Then there is a unique $u \in V$ such that

$$a(v,u) = l(v) \quad \forall v \in V, \quad \text{and } ||u|| \le \frac{\gamma}{\alpha}.$$

Note: In Lax-Milgrams theorem $a(v,u)$ need not be symmetric!

Exercises

Exercise Show the last bound for u, and the uniqueness.

Exercise Consider the problem

$$-\Delta u + \beta \cdot \nabla u = f, \quad \text{in } \Omega, \quad u = 0, \quad \text{on } \partial\Omega,$$

where $\beta = (\beta_1, \beta_2)$ is constant (or, more generally, div$\beta \le 0$). Show that Lax-Milgrams theorem applies.

Exercise Compare α, γ and $\frac{\gamma}{\alpha}$ for $\beta = 0$ with the norms $(\int_\Omega |\nabla v|^2)^{1/2}$ and $(\int_\Omega |\nabla v|^2 + v^2)^{1/2}$ on $V = H_0^1(\Omega)$, respectively.

Exercises

Consider the problem

$$-\Delta u + c\,u = 0, \quad \text{in } \Omega, \quad \partial_n u = g, \quad \text{on } \partial\Omega,$$

where $c(x) \ge c_0 > 0$. The corresponding variational problem is: Find $u \in H^1(\Omega)$ such that

$$-\int_{\partial\Omega} v\,\underbrace{\partial_n u}_{=g} + \int_\Omega (\nabla v \cdot \nabla u + c\,v\,u) = 0, \quad \forall v \in H^1(\Omega).$$

Exercise Show that Lax-Milgram can be applied, assuming, for example, that $\int_{\partial\Omega} g^2 < \infty$.

Exercises

Consider the problem

$$-\Delta u + c\,u = 0, \quad \text{in } \Omega, \quad \partial_n u = g, \quad \text{on } \partial\Omega,$$

where $c(x) \ge c_0 > 0$. The corresponding variational problem is: Find $u \in H^1(\Omega)$ such that

$$-\int_{\partial\Omega} v\,\underbrace{\partial_n u}_{=g} + \int_\Omega (\nabla v \cdot \nabla u + c\,v\,u) = \int_{\partial\Omega} v\,g, \quad \forall v \in H^1(\Omega).$$

Exercise Show that Lax-Milgram can be applied, assuming, for example, that $\int_{\partial\Omega} g^2 < \infty$.

Exercises

Exercise Show that the condition $c_0 > 0$ is nescessary to guarantee uniqueness.

Consider the problem

$$-\Delta u = f, \quad \text{in } \Omega, \quad \partial_n u + u = g, \quad \text{on } \partial\Omega.$$

The corresponding variational problem reads: Find $u \in H^1(\Omega)$ such that

$$-\int_{\partial\Omega} v\,\partial_n u + \int_\Omega (\nabla v \cdot \nabla u + c\,v\,u) = \int_\Omega v\,f, \quad \forall v \in H^1(\Omega).$$

Exercises

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$$-\int_{\partial\Omega} v\,\partial_n u + \int_\Omega (\nabla v \cdot \nabla u + c\,v\,u) + \int_{\partial\Omega} v\,u = \int_\Omega v\,f + \int_{\partial\Omega} v\,g,$$

for all $v \in H^1(\Omega)$.

Exercises

Exercise Show that

$$\int_\Omega v^2 \le C_\Omega\Big(\int_\Omega |\nabla v|^2 + \int_{\partial\Omega} v^2\Big), \quad \forall v \in H^1(\Omega).$$

Exercise Show that Lax-Milgram can be applied to the Robin problem above.

Exercises

Exercise Show that the problem

$$-\Delta u = f \quad \text{in } \Omega, \quad u = g \quad \text{on } \partial\Omega,$$

can be formulated: Find $u \in V_g$ such that

$$\int_\Omega \nabla v \cdot \nabla u = \int_\Omega v\,f, \quad \forall v \in V_0,$$

where

$$V_g = \{v : \int_\Omega |\nabla v|^2 < \infty, v = g \text{ on } \partial\Omega\},$$

or

$$F(u) = \min_{v \in V_g} F(v).$$

Lecture 6:

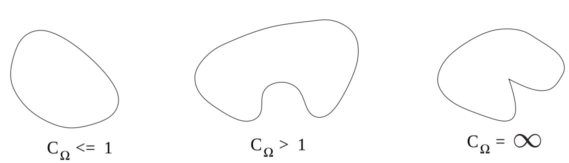
Elliptic regularity

Elliptic regularity

Can be shown: For a domain without re-entrent corners there is a constant $C = C_\Omega$ such that

$$|u|_{H^2(\Omega)} := ||D^2u||_{L_2(\Omega)} \le C_\Omega \, ||\Delta u||_{L_2(\Omega)},$$

for all u such that $u = 0$ on $\partial\Omega$ or $\partial_n u = 0$ on $\partial\Omega$, where $D^2u = (u_{xx}^2 + u_{xy}^2 + u_{yx}^2 + u_{yy}^2)^{1/2}$. For Ω *convex* we have $C_\Omega \le 1$.



Elliptic regularity, cont.

Proof (for $\Omega = \square$ and $u = 0$ on $\partial\Omega$ (homogeneous Diriclet b.c.)).
We have

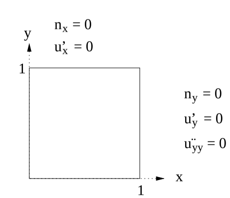
$$||\Delta u||^2 = \int_\Omega (u_{xx} + u_{yy})^2 = \int_\Omega (u_{xx}^2 + u_{yy}^2 + 2\, u_{xx}\, u_{yy}).$$

Elliptic regularity, cont.

Further

$$\int_\Omega u_{xx}\, u_{yy} = \int_{\partial\Omega} u_x\, u_{yy}\, n_x - \int_\Omega u_x \overbrace{u_{yyx}}^{=u_{xyy}}$$
$$= \underbrace{\int_{\partial\Omega} u_x\, u_{yy}\, n_x - u_x\, u_{xy}\, n_y}_{=0} + \int_\Omega u_{xy}\, u_{xy},$$

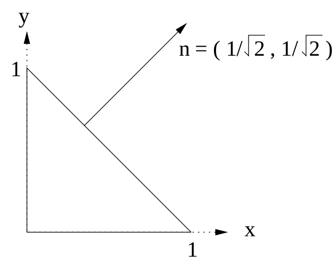
that is, here $||\Delta u|| = ||D^2 u||$.



Elliptic regularity, cont.

Exercise Show the corresponding result for boundary conditions $\partial_n u = 0$ on $\partial\Omega$.

Exercise Show the corresponding result for $\Omega = \{x : x_1 > 0, x_2 > 0, x_1 + x_2 < 1\}$:



Elliptic regularity, cont.

Exercise Show that for $\Omega = \Box$ and $\partial_n u + u = 0$ on $\partial\Omega$,

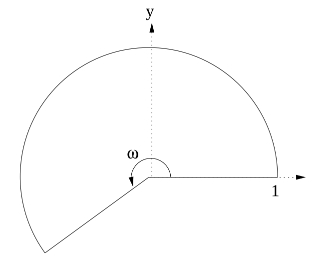
$$\int_{]} u_x\, u_{yy}\, n_x = -\int_{]} u\, u_{yy} = \int_{]} u_y^2 \geq 0,$$

and

$$||D^2 u|| \leq ||\Delta u||.$$

Corner singularities

Example Take $\Omega = \{(r,\theta) : 0 < r < 1,\, 0 < \theta < \omega\}$



and $u = r^{\frac{\pi}{\omega}}\,\mathrm{sin}(\frac{\pi}{\omega}\,\theta)$. We note that $\Delta u = \frac{1}{r}(r\,u_r')_r' + \frac{1}{r^2}u_{\theta\theta}'' = 0$, but that $D^2 u \notin L_2(\Omega)$ if $\omega > \pi$.
Exercise Verify that $u \in H^1$ for all $\omega \leq 2\,\pi$, but that $D^2 u \notin L_2$ for $\omega > \pi$, as claimed.

Corner singularities

Example $\omega = 2\,\pi \Rightarrow u = r^{\frac{1}{2}}\,\mathrm{sin}(\frac{1}{2}\theta)$, $u_r = \frac{1}{2}\,r^{-\frac{1}{2}}\,\mathrm{sin}(\frac{1}{2}\,\theta)$,
$u_{rr}'' = -\frac{1}{4}\,r^{-\frac{3}{2}}\,\mathrm{sin}(\frac{1}{2}\,\theta)$

$$\int_\Omega (u_{rr}'')^2 = \int_0^1 \int_0^{2\,\pi} \frac{1}{16}\,r^{-3}\,\mathrm{sin}^2(\frac{1}{2}\,\theta)\,r\,dr\,d\theta = \frac{\pi}{16}\int_0^1 r^{-2}\,dr = \infty.$$

Exercise Show that for solution $u = r^{\frac{\pi}{\omega}}\,(1 - r^2)\,\mathrm{sin}(\frac{\pi}{\omega}\,\theta)$ to $-\Delta u = f$ in Ω, $u = 0$ on $\partial\Omega$, we do *not* have $||D^2 u|| \leq C||f||$ for some $C < \infty$.

Elliptic regularity, cont.

More generally, one has

$$\int_{\partial\Omega} u_x\, u_{yy}\, n_x - u_x\, u_{xy}\, n_y = \int_{\partial\Omega} \partial u_n\, \partial_\tau^2 u$$

where $\partial_\tau u = \tau \cdot \nabla u$, and for $u = 0$ on $\partial\Omega$: $\partial_\tau^2 u = \partial_n u / R$, that is,

$$||D^2 u||^2 + \int_{\partial\Omega} (\partial_n u)^2 / R = ||\Delta u||^2,$$

where R is the local *radius of curvature* of $\partial\Omega$, which for Ω convex is ≥ 0 everywhere (see CDE page 525-).

Elliptic regularity, cont.

Applied to the solution of the Poisson problem $-\Delta u = f$ in Ω, $u = 0$ on $\partial\Omega$, this implies

$$||D^2 u|| \leq C_\Omega\, ||f||,$$

with, in particular, $C_\Omega \leq 1$ for Ω convex.

Space discretization with finite elements

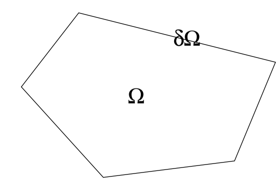
Now want to *compute* $u(x)$ such that

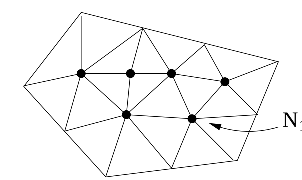
$$-\Delta u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

and recall that

$$\int_\Omega \nabla v \cdot \nabla u = \int_\Omega v\, f, \quad \forall v = v(x) \text{ with } v = 0 \text{ on } \partial\Omega.$$

We now *discretize* space by *partitioning* or *triangulating* Ω:





Discretization

We consider particular *piecewise linear* functions $\phi_i(x)$ defined on Ω, with ϕ_1 as in the figure, such that

$$\begin{cases} \phi_i(N_i) = 1, \quad \phi_i(N_j) = 0 \text{ for } j \neq i, \\ \phi_i \text{ is } \textit{linear} \text{ (or } \textit{affine}\text{) in } x \text{ on each triangle/element,} \end{cases}$$

where N_j are the *nodes* (element joints) of the triangulation. Here there are 6 interior nodes $N_1, .., N_6$ and 8 boundary nodes $(N_7, .., M_{14})$.

A discrete problem

For the computation of a solution we now make the ansatz

$$U(x) = U_1\,\phi_1(x) + U_2\,\phi_2(x) + ... + U_6\,\phi_6(x),$$

and seek coefficients $U_1, .., U_6$ such that

$$\int_\Omega \nabla\phi_j \cdot \nabla U = \int_\Omega \phi_j\, f, \quad j = 1, .., 6,$$

which is a system of 6 equations for the 6 unknowns U_i, motivated by the fact that

$$\int_\Omega \nabla v \cdot \nabla u = \int_\Omega v\, f, \quad \forall v \in H_0^1(\Omega).$$

Solving for the discrete solution U

The equations for $U(x) = U_1\,\phi_1(x) + .. + U_6\,\phi_6(x)$ reduces to

$$U_1 \int_\Omega \nabla\phi_j \cdot \nabla\phi_1 + .. + U_6 \int_\Omega \nabla\phi_j \cdot \nabla\phi_6 = \int_\Omega \phi_j\, f, \quad j = 1, .., 6.$$

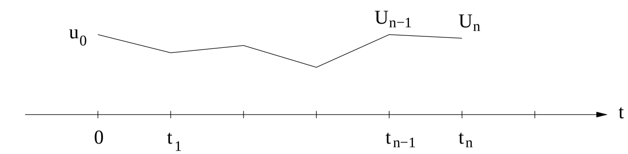
Before the coefficients $U_1, .., U_6$ can be solved for, we must calculate the coupling coefficients $\int_\Omega \nabla\phi_j \cdot \nabla\phi_k$ and $\int_\Omega \phi_j\, f$. In the corresponding one dimensional case:

The method is called cG1, for continuous Galerkin of degree 1.

Exercise See CDE 6.2, 8.1.4-6, 15.1.2-8.

Time discretization

Model problem: Seek $u(t)$ such that $\dot{u} + a\,u = f$ for $t > 0$, $u(0) = u_0$. Again we use a cG1 ansatz, seeking a piecewise linear $U(t)$ of the form:



Assume we have computed U up to time t_{n-1}, that is U_{n-1}. On the time subinterval $[t_{n-1}, t_n]$ we seek

$$U(t) = U_{n-1} \underbrace{\frac{t_n - t}{k_n}}_{\psi_{n-1}(t)} + U_n \underbrace{\frac{t - t_{n-1}}{k_n}}_{\psi_n(t)}.$$

Time discretization

To determine U_n we require that

$$\int_{t_{n-1}}^{t_n} \dot{U} + a\,U = \int_{t_{n-1}}^{t_n} f,$$

which is *one* equation for the one unknown U_n. Noting that $\dot{U} = \frac{U_n - U_{n-1}}{k_n}$ (looking like a time derivative), and assuming for simplicity that a is constant, we get

$$U_n - U_{n-1} + a\,\frac{U_{n-1} + U_n}{2}\,k_n = \int_{t_{n-1}}^{t_n} f,$$

Time discretization

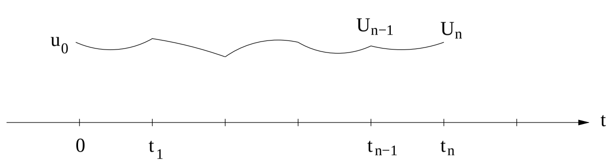
that is,

$$(1 + a\,k_n/2)\,U_n = (1 - a\,k_n/2)\,U_{n-1} + \int_{t_{n-1}}^{t_n} f,$$

from which U_n can be computed. The method is called cG1 or Crank-Nicolson.

Time discretization

A more accurate method is cG2, based on a piecewise quadratic ansatz:



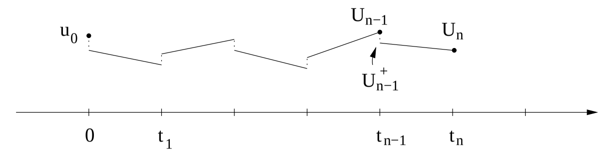
and determining U on $[t_{n-1}, t_n]$ by

$$\int_{t_{n-1}}^{t_n} v\,(\dot U + a\,U) = \int_{t_{n-1}}^{t_n} v\,f, \quad \forall v(t) \text{ of degree one,}$$

that is linear in t, with, for example, $v = \psi_{n-1}$ and $v = \psi_n$.

Time discretization

Another method uses a *dis*continuous (more flexible) ansatz called dG1:



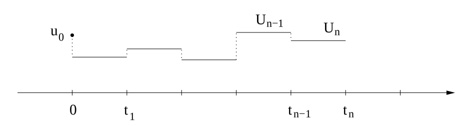
Now (like for cG2) there are two node values to determine on the new time interval:

$$v(t_{n-1})\,(U^+_{n-1} - U^-_{n-1}) + \int_{t_{n-1}}^{t_n} v\,(\dot U + a\,U) = \int_{t_{n-1}}^{t_n} v\,f, \quad \forall v(t)$$

of degree one, that is linear in t.

Time discretization

We may also consider dG0, with a piecewise *constant* ansatz for $U(t)$:



and $U_n := U(t)$ for $t_{n-1} < t \le t_n$ determined by

$$v(t_{n-1})\,(U^+_{n-1} - U^-_{n-1}) + \int_{t_{n-1}}^{t_n} v\,(\dot U + a\,U) = \int_{t_{n-1}}^{t_n} v\,f, \quad \forall v(t)$$

which are constant,reducing to just

$$(1 + a\,k_n)\,U_n = U_{n-1} + \int_{t_{n-1}}^{t_n} f.$$

Time discretization

The method is also refereed to as *implicit Euler*, while the classical or *explicit Euler* method, as we recall, is

$$U_n = (1 + a\,k_n)\,U_{n-1} + k_n\,f(t_{n-1}).$$

Lecture 7:

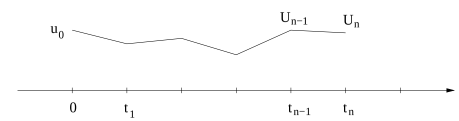
Finite element methods, cont.

FEM, cont

Have seen that solution $u(t)$ to

$$\dot{u} + a\,u = f \quad t > 0, \quad u(0) = u_0,$$

can be computed for ex. with cG1 with ansats:



and $U_n := U(t_n)$ determined by

$$\int_{I_n} \dot{U} + a\,U = \int_{I_n} f, \quad I_n = [t_{n-1}, t_n].$$

For a constant, we get

FEM, cont.

$$U_n - U_{n-1} + a\frac{U_{n-1} + U_n}{2}\,k = \int_{I_n} f, \quad k = t_n - t_{n-1},$$

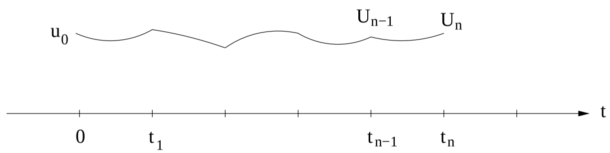
that is,

$$(1 + \frac{k}{2}\,a)\,U_n = (1 - \frac{k}{2}\,a)\,U_{n-1} + \int_{I_n} f.$$

Given $U_0 = u_0$ this gives U_1, then U_2, and so on.

FEM, cont.

For cG2 we use the ansatz

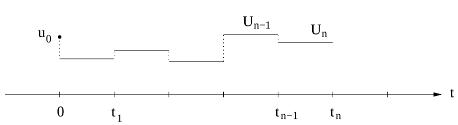


that is, U is *piecewise quadratic* in t, and U is determined by

$$\int_{I_n} v\,(\dot{U} + a\,U) = \int_{I_n} v\,f \quad \forall v = v(t) \text{ linear in } t.$$

FEM, cont.

Can also use dG0, with the ansatz of $U(t)$ as *piecewise constant*:



Here $U_n := U(t)$ for $t \in I_n$ is determined by

$$U_n - U_{n-1} + \int_{I_n} a\,U = \int_{I_n} f,$$

that is, for a constant,

$$(1 + a\,k)\,U_n = U_{n-1} + \int_{I_n} f.$$

FEM, cont.

Compare to Eulers method

$$U_n = (1 - a\,k)\,U_{n-1} + k\,f_{n-1}.$$

Example For $k = 0.1$, $a = 1$, $f = 0$, we obtain

$$U_n = 0.9\,U_{n-1} = 0.9^2\,U_{n-2} = \ldots = 0.9^n\,u_0.$$

Correspondingly for dG0:

$$U_n = \frac{1}{1.1}\,U_{n-1} = \frac{1}{1.1^2}\,U_{n-2} = \ldots$$

FEM, cont.

Compare to the exact solution formula $u(t_n) = e^{-t_n}\,u_0 = e^{-nk}\,u_0 = (e^{-k})^n\,u_0$, where the factors 0.9 and 1/1.1 above approximates $e^{-0.1}$.

Exercise Compute U_n using cG1, dG0 and Euler with $f = 0$, $k = 0.1$ and a) $a = 1$ b) $a = 30$, and compare with the exact solutions.

Exercise Generalize the methods above to a) $f(t) = t$ b) $f(t) = e^{-t^2}$ (with quadrature) c) $a = a(t) = t$.

FEM in one space dimension

Example problem: Find $u(x)$ such that

$$-u'' + u = x, \quad 0 < x < 1, \quad u(0 = 0, \quad u'(1) = 7.$$

Seek *variational formulation* ($\times v$ & $\int_0^1$)

$$\int_0^1 v(x)\, x = \int_0^1 v\,(-u'' + u) = v\,(-u')|_0^1 + \int_0^1 v'\, u' + v\, u,$$

FEM, cont.

that is,

$$\underbrace{\int_0^1 v'\, u' + v\, u}_{a(v,u)} = \underbrace{\int_0^1 v\, f + v(1)\, 7}_{l(v)} \quad \forall v(x) \text{ s.t. } v(0) = 0.$$

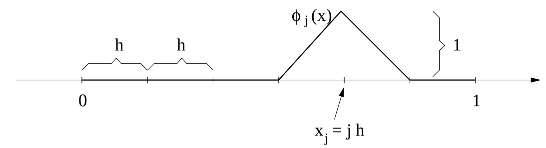
Exercise Find a variational formulation of

$-(x\, u')' = 7,\ 1 < x < 2,\ u'(1) = u(1),$ a) $u(2) = 0,$ b) $u(2) = 3.$

FEM, cont.

Discretization into finite elements

Partition the domain $[0, 1]$ into m elements/subintervals of size h:



Then seek $U(x) = U_1\, \phi_1(x) + ... + U_m\, \phi_m(x)$, that is, $U_1, .., U_m$ such that

$$\int_0^1 (\phi_j'\, U' + \phi_j\, U = \int_0^1 \phi_j\, x + \phi_j(1)\, 7, \quad j = 1, .., m.$$

FEM, cont.

This reduces to m equations for the m unknowns $U_1, .., U_m$ as follows:

$$U_1 \underbrace{\int_0^1 \phi_j'\, \phi_1' + \phi_j\, \phi_1}_{a_{j1}} + U_2 \underbrace{\int_0^1 ..}_{a_{j2}} + .. \quad = \underbrace{\int_0^1 \phi_j(x)\, x + \phi_j(1)\, 7}_{b_j},$$

that is,

FEM, cont.

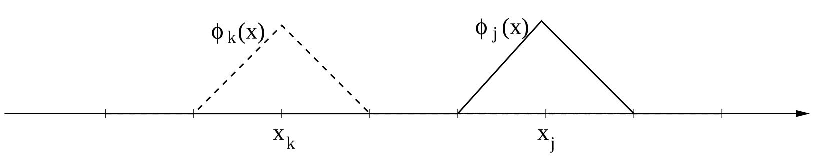
$$\underbrace{\begin{bmatrix} a_{11} & . & a_{1m} \\ . & . & . \\ a_{j1} & . & a_{jm} \\ . & . & . \\ a_{m1} & . & a_{mm} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} U_1 \\ U_2 \\ . \\ . \\ U_m \end{bmatrix}}_{U} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ b_m \end{bmatrix}}_{b} \Leftrightarrow U = A^{-1} b,$$

where

FEM, cont.

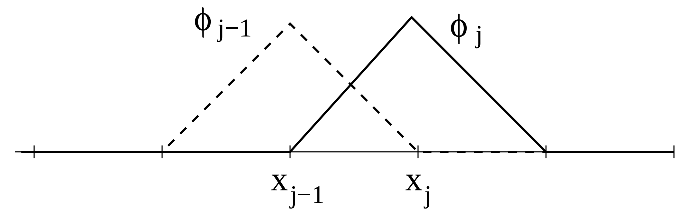
$$a_{jk} = \int_0^1 \phi'_j \, \phi'_k + \phi_j \, \phi_k .$$

We note that A is *sparse*, because



FEM, cont.

For $k = j - 1$ we obtain



$$a_{jk} = a_{j,j-1} = \int_{x_{j-1}}^{x_j} \tfrac{1}{h} \, (-\tfrac{1}{h}) + \tfrac{x - x_{j-1}}{h} \, \tfrac{x_j - x}{h} = -\tfrac{1}{h} + \tfrac{1}{6} h,$$
$$b_j = \int_0^1 \phi_j(x) \, x = x_j \int_0^1 \phi_j = h \, x_j, \quad j < m,$$

and so on..

FEM, cont.

We may summarize this as $A = S + M$ where

$$S = \frac{1}{h} \begin{bmatrix} 2 & -1 & & 0 \\ -1 & 2 & . & \\ & . & . & \\ 0 & & -1 & ? \end{bmatrix}, \quad M = h \begin{bmatrix} 2/3 & 1/6 & & 0 \\ 1/6 & 2/3 & . & \\ & . & . & \\ 0 & & 1/6 & ? \end{bmatrix},$$

$$b = h^2 \begin{bmatrix} 1 \\ 2 \\ . \\ m-1 \\ ? \end{bmatrix}$$

FEM, cont.

Finite elements in space and time

Consider the heat eq

$$\dot{u} - u'' = f, \quad u(0,t) = 0,\, u'(1,t) = 0,\, u(x,0) = u_0(x),$$

The variational formulation is now

$$\int_{I_n}\int_0^1 v\,\dot{u} + \int_{I_n}\int_0^1 v'\,u' = \int_{I_n}\int_0^1 v\,f \quad \forall v : v(0,t) = 0.$$

FEM, cont.

Note that the boundary condition at $x = 1$ is now *homogeneous*. Otherwise we would have an extra term from the integration by parts.

We now make the ansatz

$$U(x,t) = U_{n-1}\,\psi_{n-1}(t) + U_n(x)\,\psi_n(t)$$

where

$$\psi_n(t) = \frac{t - t_{n-1}}{k}$$

and

$$U_n(x) = U_{n,1}\,\phi_1(x) + U_{n,2}\,\phi_2(x) + \ldots + U_{n,m}\,\phi_m(x),$$

that is, $U(x,t)$ is piecewise linear in time and space.

FEM, cont.

The node values $U_{n,k}$ are determined by

$$\int_{I_n}\int_0^1 \phi_j\,\underbrace{\frac{U_n - U_{n-1}}{k}}_{\dot{U}} + \int_{I_n}\int_0^1 \phi_j'\,\underbrace{(U_{n-1}'\,\psi_{n-1} + U_n'\,\psi_n)}_{U'}$$

$$= \int_{I_n}\int_0^1 \phi_j\,f, \quad j = 1,..,m,$$

that is,

$$\underbrace{\int_0^1 \phi_j\,U_n}_{M\,U_n} - \underbrace{\int_0^1 \phi_j\,U_{n-1}}_{M\,U_{n-1}} + \underbrace{\int_{I_n} \psi_{n-1}}_{\frac{k}{2}}\,\underbrace{\int_0^1 \phi_j'\,U_{n-1}'}_{S\,U_{n-1}} + \underbrace{\int_{I_n} \psi_n}_{\frac{k}{2}}\,\underbrace{\int_0^1 \phi_j'\,U_n'}_{S\,U_n}$$

FEM, cont.

that is

$$(M + \frac{k}{2}\,S)\,U_n = (M - \frac{k}{2}\,S)\,U_{n-1} + F_n,$$

from which we succesively can compute $U_n = \begin{bmatrix} U_{n,1} \\ . \\ U_{n,m} \end{bmatrix}$

from U_{n-1} and F_n.

Exercise Find the corresponding formula for U_n using the dG0cG1 method (dG0 time stepping and cG1 space discretization).

FEM, cont.

Assembling

Consider the problem

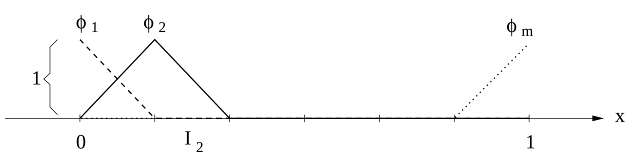
$$-u'' + u = f, \quad 0 < x < 1, \quad u'(0) = 0, \quad u'(1) = 0,$$

with variational formulation

$$\int_0^1 v'\,u' + v\,u = \int_0^1 v\,f, \quad \forall v = v(x).$$

FEM, cont.

With the ansatz $U(x) = U_1\,\phi_1(x) + .. + U_m\,\phi_m(x)$



and substituting u by U, v by ϕ_j, we obtain for $U = \begin{bmatrix} U_1 \\ . \\ U_m \end{bmatrix}$:

$$\underbrace{(S+M)}_{:=A}\,U = b,$$

where

FEM, cont.

$$S = \begin{array}{cc} \begin{bmatrix} & & \cdot & & \\ & & \cdot & & \\ \cdot & \cdot & \int_0^1 \phi_j'\,\phi_k' & \cdot & \cdot \\ & & \cdot & & \\ & & \underbrace{\cdot}_{\text{col } k} & & \end{bmatrix} & \leftarrow \text{row } j \end{array}$$

correspondingly for M and b.

We note that

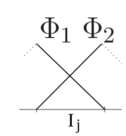
FEM, cont.

$$S = \begin{bmatrix} \int_1 & \int_1 & 0 & & 0 \\ \int_1 & \int_1 + \int_2 & \int_2 & 0 & \\ 0 & \int_2 & \int_2 + \int_3 & \int_3 & \\ & 0 & \int_3 & \int_3 + \int_4 & \\ & & & & \\ 0 & & & & \end{bmatrix}$$

Local stiffness matrix

$\int_j = \int_{I_j}$

$\int_3 \Phi_1'\,\Phi_2'$



with local stiffness matrix dS defined by

FEM, cont.

$$dS_3 = \begin{bmatrix} \int_{I_3} \phi_3' \phi_3' & \int_{I_3} \phi_3' \phi_4' \\ \int_{I_3} \phi_4' \phi_3' & \int_{I_3} \phi_4' \phi_4' \end{bmatrix} = \begin{bmatrix} \int_{I_3} \backslash' \backslash' & \int_{I_3} \backslash' /' \\ \int_{I_3} /' \backslash' & \int_{I_3} /' /' \end{bmatrix},$$

where

that is, Φ_1, Φ_2 *local basis* for the linears on I_3.

FEM, cont.

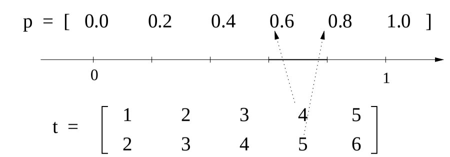
Leads to *assembling algorithm* as follows:

```
for el = 1 :  m-1
  dS(el) = ...
  adding dS(el) to S
end
```

Similarly for M and b.

Computation of dS(el)

Data structures for node coords and element-node couplings:



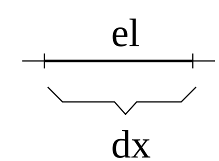
The columns in p holds the coordinates of the different nodes. The columns in t gives the node numbers of the different elements. For example, element 4 has the nodes 4 and 5. The coordinates for these are $p(4) = p(t(1,4))$ and $p(5) = p(t(2,4))$. In general the node coordinates for element el is given by

$$x = p(t(:, el)).$$

Computation of dS(el), cont.

The integrals in $dS(el)$ are given by the element area

$$dx = x(2) - x(1) = p(t(2, el)) - p(t(1, el)).$$



and the derivatives of the local basis functions Φ_1 and Φ_2, that is

$$DPhi = [-1/dx, 1/dx],$$

by the computation

Computation of dS(el), cont.

$$dS(el) = DPhi^\top * DPhi * dx = \begin{bmatrix} -1/dx \\ 1/dx \end{bmatrix} [-1/dx \ 1/dx]\, dx$$

$$= \begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix} \frac{1}{dx}.$$

The corresponding integrals in dM(el) and db(el) must be computed using suitable quadrature rules, like the trapetzoidal rule or midpoint rule.

Adding dS(el) to S

The row- and column indices of S of which the elements of $dS = dS(el)$ are a part are given by $t(:,el)$, that is, S can be updated with the contributions from element el with

$$S(r,k) = S(r,k) + dS(el),$$

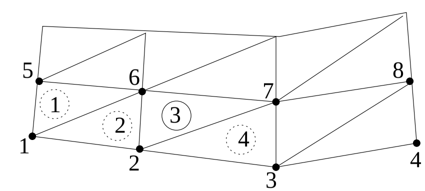
where $r = k = t(:,el)$. For example, for element $el = 3$, that is I_3, we have $t(:,el) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, and $dS(el)$ is added to

$\begin{bmatrix} S_{33} & S_{34} \\ S_{43} & S_{44} \end{bmatrix}$ in S.

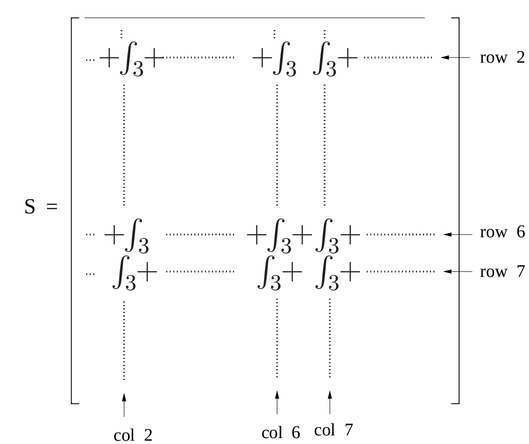
2D assembling

Similarly for the problem

$$-\Delta u + u = f \quad \text{in } \Omega, \quad \partial_n u = 0, \quad \text{on } \partial\Omega,$$



2D assembling

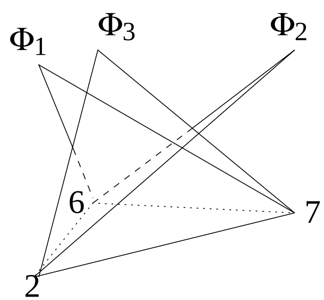


with the local stiffness matrix

2D assembling

$$dS_3 = \begin{bmatrix} \int_{K_3} \nabla\Phi_1 \cdot \nabla\Phi_1 & \int_{K_3} \nabla\Phi_1 \cdot \nabla\Phi_2 & \int_{K_3} \nabla\Phi_1 \cdot \nabla\Phi_3 \\ \int_{K_3} \nabla\Phi_2 \cdot \nabla\Phi_1 & \int_{K_3} \nabla\Phi_2 \cdot \nabla\Phi_2 & \int_{K_3} .. \\ \int_{K_3} \nabla\Phi_3 \cdot \nabla\Phi_1 & \int_{K_3} .. & \int_{K_3} .. \end{bmatrix}$$

where



2D assembling

Corresponding data structures p and t are now

$$p = \begin{bmatrix} . & p(1,2) & . & . & p(1,6) & p(1,7) & . \\ . & p(2,2) & . & . & p(2,6) & p(2,7) & . \end{bmatrix} \begin{matrix} \leftarrow x_1 - \text{ccords} \\ \leftarrow x_2 - \text{ccords} \end{matrix}$$

$$t = \begin{bmatrix} . & . & 2 & . & .. \\ . & . & 7 & . & .. \\ . & . & 6 & . & .. \end{bmatrix}$$

where element 3 has the nodes 2, 7 and 6, whos x_1 and x_2 coordinates can be found in p.
Note that the ordering 2, 7, 6 defines a *local node-ordering* of the nodes of the element to correspond to the local basis functions Φ_1, Φ_2 and Φ_3 as in the figure.

2D assembling

The element length dx is now replace by the element *area* dx, and the basis function "derivative vector" by the basis function *gradient matrix*

$$DPhi = [\nabla\Phi_1 \quad \nabla\Phi_2 \quad \nabla\Phi_3], \quad \text{where } \nabla\Phi_j = \begin{bmatrix} \frac{\partial\Phi_j}{\partial x_1} \\ \frac{\partial\Phi_j}{\partial x_2} \end{bmatrix},$$

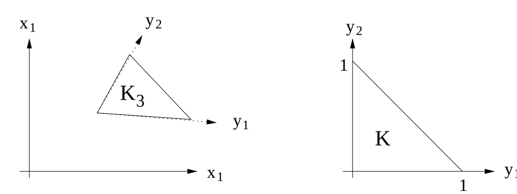
while the computation of $dS(el)$ and the adding of dS to S look exactly as before.

Reference element

More generally, the local stiffness matrices (and load vectors) are often computed on a *reference element* K, after coordinate transformation $x = x(y)$, for example

$$\int_{K_3} \Phi_i \, \Phi_j \, dx = \int_K \hat{\Phi}_i \, \hat{\Phi}_j \, |\frac{dx}{dy}| \, dy,$$

where $\hat{\Phi}_i(y) = \Phi_i(x(y))$, $K_3 = \{x(y) : y \in K\}$, and $|\frac{dx}{dy}|$ denotes the Jacobian.



Lecture 8:

FEM for the wave equation

FEM for the wave eq.

We now seek $u(x,t)$ such that

$$\begin{cases} \ddot{u} - u'' = 0, \quad 0 < x < 1, \quad t > 0, \\ u(0,t) = 0, \quad u'(1,t) = g(t), \quad t > 0, \\ u(x,0) = u_0(x), \quad \dot{u}(x,0) = \dot{u}_0(x), \quad 0 < x < 1. \end{cases}$$

Rewrite as *system* of eq's:

$$\begin{cases} \dot{u} - v = 0, \\ \dot{v} - u''. \end{cases}$$

Exercise Write this system on matrix form as $\dot{w} + a\,w = 0$ with $w = \begin{bmatrix} u \\ v \end{bmatrix}$.

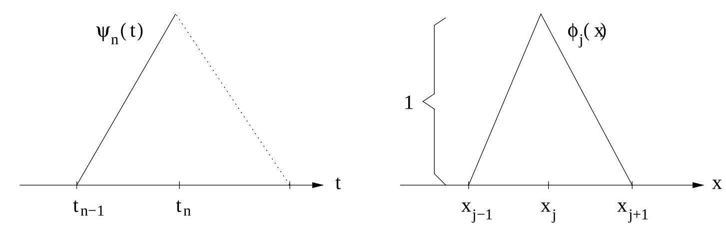
FEM for the wave eq., cont.

We now make ansatz

$$U(x,t) = U_{n-1}(x)\,\psi_{n-1}(t) + U_n(x)\,\psi_n(t), \quad t \in I_n$$
$$V(x,t) = V_{n-1}(x)\,\psi_{n-1}(t) + ...$$

where

$$U_n(x) = U_{n,1}\,\phi_1(x) + U_{n,2}\,\phi_2(x) + ... + U_{n,m}\,\phi_m(x) \quad \text{etc}$$



FEM for the wave eq., cont.

We note that $\dot{u} - v = 0$ for $t \in I_n$ $\Leftrightarrow$

$$\int_{I_n}\int_0^1 \phi\,\dot{u} - \int_{I_n}\int_0^1 \phi\,v = 0 \quad \text{for all } \phi(x,t),$$

and that $\dot{v} - u'' = 0$ & $u'(1,t) = g(t)$ $\Leftrightarrow$

$$\int_{I_n}\int_0^1 \phi\,\dot{v} + \int_{I_n}\int_0^1 \phi'\,u' - \int_{I_n} \phi\,u'|_{x=0}^{x=1} = 0,$$

for all ϕ.

FEM for the wave eq., cont.

We note that $\dot u - v = 0$ for $t \in I_n \Leftrightarrow$

$$\int_{I_n}\int_0^1 \phi\,\dot u - \int_{I_n}\int_0^1 \phi\, v = 0 \quad \text{for all } \phi(x,t),$$

and that $\dot v - u'' = 0$ & $u'(1,t) = g(t) \Leftrightarrow$

$$\int_{I_n}\int_0^1 \phi\,\dot v + \int_{I_n}\int_0^1 \phi'\,u' - \int_{I_n} \phi\,u'|_{x=0}^{x=1} = \int_{I_n} \phi(1,t)\,g(t),$$

for all ϕ such that $\phi(0,t) = 0$.
We therefore seek $U(x,t)$ & $V(x,t)$ as above such that

FEM for the wave eq., cont.

$$\int_{I_n}\int_0^1 \phi_j\,\underbrace{\frac{U_n - U_{n-1}}{k}}_{\dot U} - \int_{I_n}\int_0^1 \phi_j\,\underbrace{(V_{n-1}\psi_{n-1} + V_n\psi_n)}_{V} = 0,$$

and

$$\int_{I_n}\int_0^1 \phi_j\,\underbrace{\frac{V_n - V_{n-1}}{k}}_{\dot V} + \int_{I_n}\int_0^1 \phi_j'\,\underbrace{(U_{n-1}'\psi_{n-1} + U_n'\psi_n)}_{U'}$$
$$= \int_{I_n} \phi_j(1)\,g(t), \quad j = 1,..,m.$$

This reduces to

FEM for the wave eq., cont.

$$\underbrace{\int_0^1 \phi_j\,U_n}_{MU_n} - \frac{k}{2}\underbrace{\int_0^1 \phi_j\,V_n}_{MV_n} = \int_0^1 \phi_j\,U_{n-1} + \frac{k}{2}\int_0^1 \phi_j\,V_{n-1},$$

and

$$\underbrace{\int_0^1 \phi_j(x)\,V_n(x)}_{MV_n} + \frac{k}{2}\underbrace{\int_0^1 \phi_j'(x)\,U_n'(x)}_{SU_n} = \int_0^1 \phi_j\,V_{n-1} - \frac{k}{2}\int_0^1 \phi_j'\,U_{n-1}'$$
$$+ \underbrace{\int_{I_n} \phi_j(1)\,g(t)}_{g_n}, \quad j = 1,..,m,$$

where

FEM for the wave eq., cont.

$$S = \frac{1}{h}\begin{bmatrix} 2 & -1 & & & 0 \\ -1 & 2 & -1 & & \\ & -1 & 2 & . & \\ & & . & . & . \\ 0 & & & -1 & 1 \end{bmatrix}, \quad M = h\begin{bmatrix} 2/3 & 1/6 & & 0 \\ 1/6 & 2/3 & . & \\ & . & . & \\ 0 & & 1/6 & 1/3 \end{bmatrix}$$

$$U_n = \begin{bmatrix} U_{n,1} \\ U_{n,2} \\ . \\ . \\ U_{n,m} \end{bmatrix}, \quad g_n = \begin{bmatrix} 0 \\ . \\ . \\ 0 \\ g_{n,m} \end{bmatrix}, \quad g_{n,m} = \int_{I_n} g(t).$$

FEM for the wave eq., cont.

That is, U_n and V_n is determined by

$$\begin{cases} MU_n - \frac{k}{2}MV_n = MU_{n-1} + \frac{k}{2}MV_{n-1} \\ \frac{k}{2}SU_n + MV_n = -\frac{k}{2}SU_{n-1} + MV_{n-1} + g_n. \end{cases}$$

System of $2m$ equations and unknowns:

$$\underbrace{\begin{bmatrix} M & -\frac{k}{2}M \\ \frac{k}{2}S & M \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} U_n \\ V_N \end{bmatrix}}_{W} = \underbrace{\begin{bmatrix} M & \frac{k}{2}M \\ -\frac{k}{2}S & M \end{bmatrix} \begin{bmatrix} U_{n-1} \\ V_{n-1} \end{bmatrix} + \begin{bmatrix} 0 \\ g_n \end{bmatrix}}_{b}$$

where $W = A\backslash b$, $U_n = W(1:m)$ and $V_n = W(m+1:2*m)$.

FEM for the wave eq., cont.

Exercise Derive the corresponding system for dG0 time stepping.

Exercise Show that the cG1cG1 method for the wave equation on system form with $g = 0$ is energy conserving:

$$||V_n||^2 + ||U_n'||^2 = ||V_{n-1}||^2 + ||U_{n-1}'||^2.$$

Hint: multiply one of the equations with $(U_{n-1} + U_n)^\top SM^{-1}$ and the other with $(V_{n-1} + V_n)^\top$ and add the two. Use the fact that $U_n^\top SU_n = ||U_n'||^2$, where $U_n = [U_{n,1}, .., U_{n,m}]^\top$ and $U_n = U_n(x) = U_{n,1}\phi_1(x) + .. + U_{n,m}\phi_m(x)$, respectively.

FEM for the wave eq., cont.

Exercise According to previous result, cG1 applied to $\dot{w} + a\,w = 0$ as above, leads to

$$W_n(x) - W_{n-1}(x) + \frac{k}{2}\,a\,(W_{n-1}(x) + W_n(x)) = 0,$$

where $W_n = \begin{bmatrix} U_n \\ V_n \end{bmatrix}$. Verify that a cG1 space discretization of this equations leads to the same system as above.

FEM for the wave eq., cont.

Exercise Apply cG1 time stepping directly to the wave equation, that is, using the ansatz

$$U(x,t) = U_{n-1}(x)\,\psi_{n-1}(t) + U_n(x)\,\psi_n(t),$$

and noting that $\dot{U}$ is piecewise constant in time, interprete

$$\underbrace{\int_{I_n}\int_0^1 \phi_j\,\ddot{U}}_{?} + \underbrace{\int_{I_n}\int_0^1 \phi_j'\,U'}_{\frac{k}{2}S(U_{n-1}+U_n)} = \underbrace{\int_{I_n} \phi_j(1)\,g(t)}_{g_n},\ \ j = 1,..,m.$$

FEM for Poisson equations

Exercise Show that FEM with mesh size h for the problem $-u'' = f$ for $0 < x < 1$, $u(0) = 7$, $u'(1) = 0$, with the ansatz

$$U(x) = 7\phi_0(x) + U_1\phi_1(x) + .. + U_m\phi_m(x),$$

give rise to the system of equations

FEM for Poisson equations, cont.

$$\tilde{A}\tilde{U} = \tilde{b},$$

where

$$\tilde{A} = \frac{1}{h}\underbrace{\begin{bmatrix} -1 & 2 & -1 & & 0 \\ 0 & -1 & 2 & -1 & \\ & & . & . & . \\ 0 & 0 & & -1 & 2 \end{bmatrix}}_{m\times(m+1)}, \quad \tilde{U} = \underbrace{\begin{bmatrix} 7 \\ U_1 \\ . \\ . \\ U_m \end{bmatrix}}_{(m+1)\times 1}, \quad \tilde{b} = \underbrace{\begin{bmatrix} h \\ . \\ h \\ h/2 \end{bmatrix}}_{m\times 1},$$

which may then be reduced to $AU = b$ with

FEM for Poisson equations, cont.

$$A = \frac{1}{h}\begin{bmatrix} 2 & -1 & & 0 \\ -1 & 2 & -1 & \\ & . & . & . \\ 0 & & -1 & 2 \end{bmatrix}, \quad U = \begin{bmatrix} U_1 \\ . \\ . \\ U_m \end{bmatrix}, \quad b = \begin{bmatrix} h + 7/h \\ h \\ . \\ h/2 \end{bmatrix},$$

FEM for the wave eq., cont.

Exercise Formulate a FEM for the problem

$$\begin{cases} \ddot{u} + \dot{u} - u'' = f \quad 0 < x < 1, \quad t > 0 \\ u(0,t) = 0, \quad u'(1,t) = 0, \quad t > 0 \\ u(x,0) = 0, \quad \dot{u}(x,0) = 0, \quad 0 < x < 1. \end{cases}$$

Exercise Assume $u = u(x)$ satisfies

$$\int_0^1 v'\,u' = \int_0^1 v\,f \quad \text{for all } v(x) \text{ such that } v(0) = 0.$$

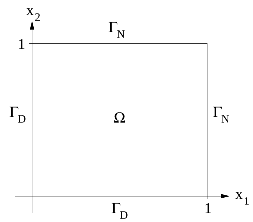
Show that $-u'' = f$ for $0 < x < 1$, and that $u'(1) = 0$.

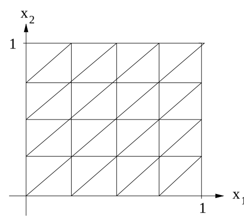
FEM for Poisson equations, cont.

Exercise Consider the problem

$$-\Delta u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \Gamma_D, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma_N,$$

with notation as in figure, with also a *trangulation* of Ω.

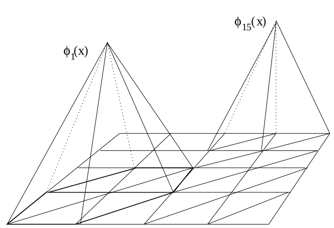




FEM for Poisson equations, cont.

Put $U(x) = U_1\,\phi_1(x) + .. + U_{16}\,\phi_{16}(x)$, with $x = (x_1, x_2)$ and *basis functions* ϕ_j as in figure, and determine the node values $U_i, i = 1, .., 16$ from

$$\int_\Omega \nabla\phi_j \cdot \nabla U = \int_\Omega \phi_j\, f, \quad j = 1, .., 16.$$



FEM for Poisson equations, cont.

Exercise Generalize to

$$-\nabla(a\nabla u) = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega_D, \; a\frac{\partial u}{\partial n} = 7 \quad \text{on } \partial\Omega_N,$$

where $a = 1$ for $x_1 < 0.5$, $a = 2$ for $x_1 > 0.5$, $f = x_2$, with mesh size h.

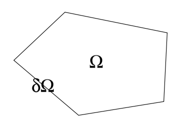
Lecture 9:

Error estimates

Recall: Variational formulation

For the Poisson problem

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

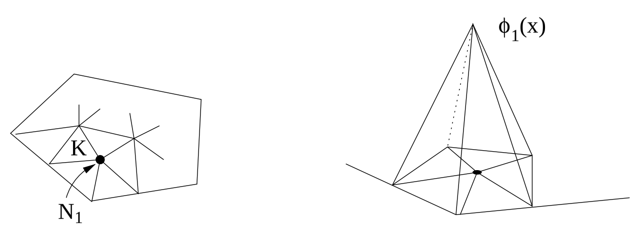


where $\Omega \subset R^d$, $d = 1, 2, 3$, with the *variational formulation*:
Find u such that $u = 0$ on $\partial\Omega$ and

$$\int_\Omega \nabla v \cdot \nabla u = \int_\Omega v\, f$$

Recall: FEM

T *triangulation* of Ω, $T = \{K : \cup K = \Omega\}$, ϕ_j, $j = 1, .., m$ corresponding *basis functions* such that ϕ_j continuous on Ω, linear in $x = (x_1, .., x_d)$ on each K, and $\phi_j(N_i) = 1$ for $i = j$, $\phi_j(x_i) = 0$ otherwise, where $N_1, .., N_m$ are the interior nodes of T



Recall: FEM, cont.

We then make the ansatz $U(x) = U_1\phi_1(x) + ... + U_m\phi_m(x)$ and seek $U_1, .., U_m$ such that

$$\int_\Omega \nabla\phi_j \cdot \nabla U = \int_\Omega \phi_j\, f, \quad j = 1, .., m,$$

or

The error equation

$$(V_h) \qquad \int_\Omega \nabla v \cdot \nabla U = \int_\Omega v\, f, \quad \text{for all } v \in V_h,$$

where

$$V_h = \{v(x) : v \text{ cont. on } \Omega, \text{ p.w. lin. w.r.t. } T, v = 0 \text{ on } \partial\Omega\}.$$

Note: For $v \in V_h$ we have
$v(x) = v(N_1)\,\phi_1(x) + .. + v(N_m)\,\phi_m(x)$.

For the *error* $e = u - U$ we have (take (V)-(V_h)):

$$\int_\Omega \nabla v \cdot \nabla e = 0 \quad \text{for all } v \in V_h.$$

Analysis

From above we get

$$||\nabla e||^2 = \int_\Omega \nabla(u-U)\cdot\nabla e = \int_\Omega \nabla u\cdot\nabla e - \underbrace{\int_\Omega \nabla U\cdot\nabla e}_{=0}$$

$$= \int_\Omega \nabla u\cdot\nabla e - \underbrace{\int_\Omega \nabla v\cdot\nabla e}_{=0} = \int_\Omega \nabla(u-v)\cdot\nabla e$$

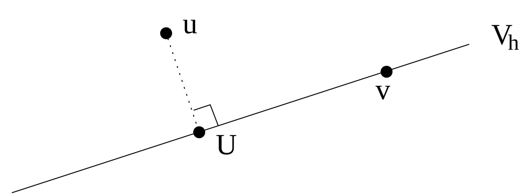
$$\le ||\nabla(u-v)||\,||\nabla e||,$$

that is

$$||\nabla e|| \le ||\nabla(u-v)|| \quad \text{for all } v\in V_h.$$

Error estimates: Error analysis

We conclude that the error e is *orthogonal* to V_h:

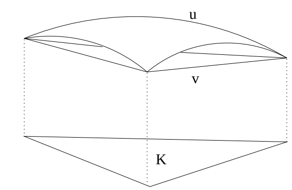


It can further be shown that there exists $v\in V_h$ such that

FEM for the wave eq., cont.

$$||\nabla(u-v)|| \le C||hD^2u||,$$

where $h=h(x)=$ diam$K$ for $x\in K$ is the local mesh size, C is a constant independent of h (and u) and D^2u is a bound for the maximal second derivative of u. This estimate is valid, for example for the *interpolant* of u, which takes the same values as u in the nodel points N_i:

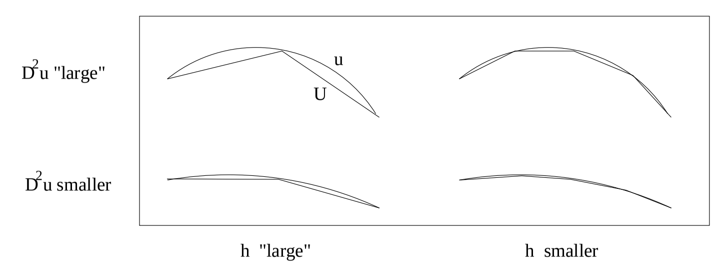


FEM for the wave eq., cont.

We thus have

$$||\nabla(u-U)|| \le C||hD^2u||,$$

which shows that the error is small if h is sufficiently small (depending on D^2u):



An estimate for $u - U$

Let ϕ be the solution to the *dual* problem (with $e = u - U$)

$$-\Delta\phi = e \quad \text{in } \Omega, \qquad \phi = 0 \quad \text{on } \partial\Omega.$$

Then

$$\begin{aligned} ||e||^2 &= \textstyle\int_\Omega (-\Delta\phi)\, e = \int_\Omega \nabla\phi \cdot \nabla e = \int_\Omega \nabla(\phi - v) \cdot \nabla e \\ &\leq ||\nabla(\phi - v)||\, ||\nabla e|| \quad \text{for all } v \in V_h. \end{aligned}$$

Now choose v such that

$$||\nabla(\phi - v)|| \leq C||hD^2\phi|| \leq C\text{max}_\Omega h||D^2\phi||,$$

and use the fact that

$$||D^2\phi|| \leq C_\Omega||\Delta\phi|| = C_\Omega||e||.$$

An estimate for $u - U$, cont.

We may now conclude, because $||\nabla e|| \leq C||hD^2u||$, that

$$||u - U|| \leq C\text{max}_\Omega h||hD^2u|| \quad \text{(an extra } h\text{)}.$$

All of our error estimates so far has been *a priori* error estimates, giving bounds for the error *a priori* (before) the computation of U. They are of a *qualitative* nature.

An a posteriori error estimate

We estimate $u - U$ and consider for simplicity the case $d = 1$. As above $-\phi'' = e$ in Ω and we get

$$\begin{aligned} ||e||^2 &= \textstyle\int_\Omega (\phi - v)'\, e' = \int_\Omega (\phi - v)\, \underbrace{(-e'')}_{=:r} \\ &\leq ||h^{-2}(\phi - v)||\, ||h^2\, r|| \leq ||\phi''||\, ||h^2\, r||, \end{aligned}$$

where we assumed that $v \in V_h$ interpolant of ϕ, and $r = -(u - U)'' = -u'' + U'' = f + U''$, that is

An a posteriori error estimate

$$||u - U|| \leq C||h^2\, r||,$$

where $r = f + U''$ is the *residual*, which is a *computable* quantity (once U has been computed).

Here, in fact, $r = f$, because $U'' = 0$ on each element K for $d = 1$.

A posteriori error estimates

Exercise Show that $||(u-U)'|| \le C||hr||$.

Exercise Verify that for the interpolant v of ϕ one has

$$||h^{-2}(\phi - v)|| \le C||\phi''||,$$

and

$$||h^{-1}(\phi - v)|| \le C||\phi'||.$$

Exercise Derive the corresponding a posteriori error estimate for $d=2$. Note that $\nabla e\,(\phi - v)$ now cannot be assumed to vanish along the element interfaces, as is the case for $d=1$. See also CDE.

Lecture 10:

Error analysis, cont.

Error analysis, cont.

Consider the heat conduction problem

$$\begin{cases} \dot{u} - u'' = f & \text{in } \Omega = (0,1), \quad t > 0, \\ u = 0 & \text{on } \partial\Omega, \quad t > 0, \\ u = u_0 & \text{in } \Omega, \quad t = 0. \end{cases}$$

The approximate solution
$U(x,t) = U_{n-1}(x)\,\psi_{n-1}(t) + U_n(x)\,\psi_n(t)$ with
$U_n(x) = U_{n,1}\,\phi(x) + .. + U_{n,m}\,\phi_m(x)$ is determined by

$$\int_{I_n}\int_\Omega \dot{U}\,\phi_j + U'\,\phi_j' = \int_{I_n}\int_\Omega f\,\phi_j \quad j = 1,..,m,$$

or, equivalently,

Error analysis, cont.

$$\int_{I_n}\int_\Omega \dot{U}\,v + U'\,v' = \int_{I_n}\int_\Omega f\,v \text{ for all } v \in V_h,$$

where

$V_h = \{v(x) : v$ cont. on $[0,1],$ p.w. linear, and $v(0) = v(1) = 0\}$.

We recall that the exact solution u satisfies these equations (for *all* $v(x,t)$ such that $v(0,t) = v(1,t) = 0$, in fact) so that, in particular, for the *error* $e = u - U$ we have

$$\int_{I_n}\int_\Omega \dot{e}\,v + e'\,v' = 0 \quad \text{for all } v \in V_h.$$

Let now $\phi)x,t)$ be the solution of the *dual* problem

Error analysis, cont.

$$\begin{cases} -\dot\phi - \phi'' = 0 & \text{in } \Omega, \quad t>0, \\ \phi = 0 & \text{on } \partial\Omega, \quad t>0, \\ \phi = e & \text{in } \Omega, \quad \text{for } t = T, \end{cases}$$

where $e = e(\cdot,T) = u(\cdot,T) - U(\cdot,T)$, $T = t_N$. Note that for $w(x,t) = \phi(x,T-t)$ we have

$$\begin{cases} \dot w - w'' = 0 & \text{in } \Omega, \quad t>0, \\ w = 0 & \text{on } \partial\Omega, \quad t>0, \\ w = e & \text{in } \Omega, \quad \text{for } t = 0, \end{cases}$$

Error analysis, cont.

Have seen that

$$\int_\epsilon^T ||\dot w|| \le \sqrt{\log(\frac{T}{\epsilon})}\, ||e||,$$

that is,

$$\int_0^{T-\epsilon} ||\dot\phi|| \le \sqrt{\log(\frac{T}{\epsilon})}\, ||e||,$$

and

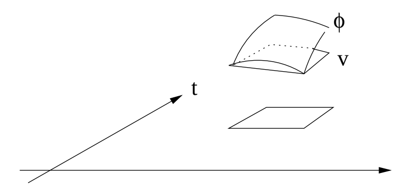
$$\int_0^{T-\epsilon} ||\phi''|| \le \sqrt{\log(\frac{T}{\epsilon})}\, ||e||.$$

Error analysis, cont.

Vi now have (assuming $u_0 \in V_h$ for simplicity)

$$||e||^2(T) = \int_\Omega e(T)\, e(T) + \int_0^T \int_\Omega \underbrace{(-\dot\phi - \phi'')}_{=0}\, e$$
$$= \int_0^T \int_\Omega \phi\, \dot e + \phi'\, e' = \int_0^T \int_\Omega (\phi - v)\, \dot e + (\phi' - v')\, e'$$
$$= \int_0^T \int_\Omega \underbrace{(\phi - v)}_{\text{“}\le k\dot\phi + h^2\phi''\text{“}} \underbrace{(f - \dot U + U'')}_{=:r}$$

Here $h = h(x,t)$ and $k = k(t)$ (or possibly $k = k(x,t)$).



Error analysis, cont.

Essentially, this reduces to

$$||e(T)||^2 \le \int_0^T (||\dot\phi|| + ||\phi''||)\, ||(k+h^2)\, r||$$
$$\le \underbrace{\Big(\int_0^{T-\epsilon} ||\dot\phi|| + ||\phi''|| + \max_{[T-\epsilon,T]} ||\phi||\Big)}_{\le \sqrt{\log\frac{T}{\epsilon}}\, ||e(T)||} \max_{[0,T]} ||(k+h^2)\, r||$$

giving

$$||e(T)|| \le \sqrt{\log(\frac{T}{\epsilon})}\, \max_{[0,T]} ||(k+h^2)\, r||.$$

Adaptivity

Starting point is an *a posteriori* error estimate for $e = u - U$, for example for $-\Delta u = f$ in $\Omega \subset R^2$, $u = 0$ on $\partial\Omega$, we have

$$||\nabla e|| \leq C||h\, r(U)||,$$

where $r(U) = |f| + \max_{\partial K}|[\nabla U]|$, where [] denotes the (discontinuity) jump.

Algorithm:

- take (arbitrary) $h = h(x)$
- given h, compute corresponding U
- accept U if $C||hr(U)|| \leq$ tol, otherwise find a new more suitable $h = h(x)$ and return to 2.

Adaptivity, cont.

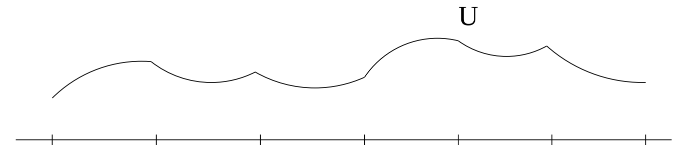
Often, the a posteriori error estimate involves a *dual* problem. For example

$$|e(x_0)| \leq \int_\Omega |D^2\phi|\, h^2\, |r(U)|,$$

where $-\Delta\phi = \delta(\cdot - x_0)$ in Ω, $\phi = 0$ on $\partial\Omega$.

Higher order elements

$cG2$ means using piecewise *quadratic* polynomials:



determined by the endpoint values and midpoint values on each subinterval, and can therefore be built from the basis functions

Higher order elements

We now have the following error estimates:

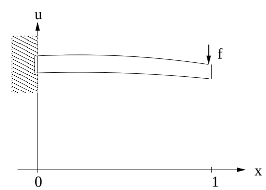
- $||(u - U)'|| \leq C||h^2 D^3 u||$
- $||u - U|| \leq C||h^3 D^3 u||$
- $||u - U|| \leq C||h^2 r(U)||$ where now $|r(U)| \leq Ch$,

for $-u'' = f$, $0 < x < 1$, with boundary conditions.

The beam equation

$$\begin{cases} (au'')'' = f \quad \text{in } \Omega = (0,1), \\ u(0) = 0 \text{ \& } u'(0) = 0 \quad \text{(Dirichlet type b.c.)} \\ au''(1) = 0 \text{ \& } (au'')'(1) = 0 \quad \text{(Neumann type b.c.)} \end{cases}$$

Here a is the bending “stiffness”, au'' is the bending *moment*, f is the load, and $u = u(x)$ is the deflection.

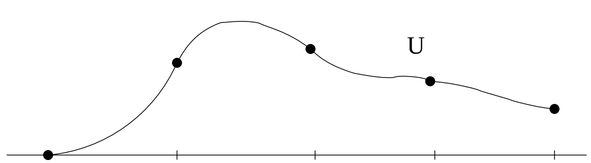


The beam eq., variational formulation

$$\int_0^1 v''\,au'' = \int_0^1 v\,f \quad \text{for all } v(x) \text{ such that } v(0) = 0 \text{ \& } v'(0) = 0.$$

Finite element method

Piecewise linear and also (!) quadratics are no good (check this!). Instead we work with piecewise *cubic* polynomials, with *continuous derivative*:



Cubic spline elements

Place two degrees of freedom in each node: The value $v(x)$ and the derivative $v'(x)$. Note that a cubic polynomial $v(x)$ on $[a,b]$ is determined by $v(a)$, $v'(a)$, $v(b)$ and $v'(b)$. As basis funtions we use

Exercise Set up the discrete system of equations for the beam problem above with $a = 1$, $f = \delta(x-1)$ and suitable meshsize h. Solve and plot the solution.

Interpolation, exercises

Exercise Assume $x, \bar{x} \in I := [a,b]$ & $w(\bar{x}) = 0$. Show that

$$|w(x)| \leq \int_I |w'|.$$

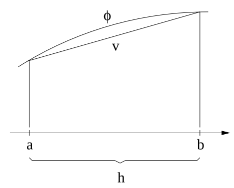
Exercise Assume v interpolates ϕ in a and b. Show that

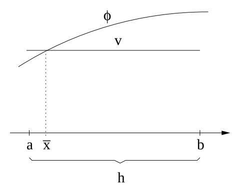
$$|(\phi - v)\,(x)| \leq \int_I |(\phi - v)'|$$

and

$$|(\phi - v)'(x)| \leq \int_I |(\phi - v)''| = \int_I |\phi''|.$$

Interpolation, exercises





Interpolation, exercises

Exercise Show further that

$$\max_I|\phi - v| \leq \max_I|h^2\phi''|$$

$$\int_I |\phi - v| \leq \int_I |h^2\phi''|$$

$$||\phi - v||_I \leq ||h^2\phi''||_I \text{ \& } ||h^{-2}(\phi - v)||_I \leq ||\phi''||,$$

where $||w|| = (\int_I w^2)^{1/2}$.

Interpolation exercises

Exercise Use that

$$v' = \frac{\phi(b) - \phi(a)}{h} = \frac{1}{h}\int_a^b \phi' \quad (= \text{const. on } I),$$

to show that

$$|(\phi - v)\,(x)| \leq 2\int_I |\phi'|,$$

and thus

$$\int_I h^{-1}|\phi - v| \leq 2\int_I |\phi'| \text{ \& } ||h^{-1}(\phi - v)||_I \leq 2||\phi'||_I.$$

Interpolation exercises

Exercise Assume now that $v = v(t)$ is a constant interpolant of $\phi(t)$ on I, as above: Show that

$$\int_I h^{-1}|\phi - v| \leq \int_I |\phi'|.$$

Exercise Show that if $\dot{u} + u = f$, $t > 0$, $u(0) = u_0$ and U is the cG1-approximation of u, then

$$|(u - U)(T)| \leq \max_{[0,T]}|k(f - \dot{U} - U)|.$$

Solution:

Interpolation exercises

We start from the *error equation*

$$\int_0^T v\,(\dot e + e) = 0, \quad (e = u - U)$$

here valid for all $v = v(t)$ which are piecewise *constant*.
With ϕ such that

$$-\dot\phi + \phi = 0 \quad \text{for } t < T, \quad \phi(T) = e(T),$$

it follows that

Interpolation, exercises

$$|e(T)|^2 = \phi(T)\,e(T) + \int_0^T \underbrace{(-\dot\phi + \phi)}_{=0}\,e$$

$$= \int_0^T \phi\,(\dot e + e) = \int_0^T (\phi - v)\,(\dot e + e) = \int_0^T (\phi - v)\,\underbrace{(f - \dot U - U)}_{=:r}$$

$$\leq \int_0^T k^{-1}|\phi - v|\,\max_{[0,T]}|k\,r(U)|.$$

Interpolation exercises

Using our above interpolation error estimate and the fact that $\dot\phi = \phi$ & $\phi(t) = \exp(t - T)\,e(T)$, we conclude that

$$|e(T)|^2 \leq \underbrace{\int_0^T \exp(t - T)\,dt}_{=1-\exp(-T)\leq 1}\ \max_{[0,T]}|k\,r(U)|,$$

that is

$$|e(T)| \leq \max_{[0,T]}|kr|.$$

Interpolation exercises

Exercise Generalize to the case $\dot u + au = f$, with a constant > 0.

Exercise Is the a posteriori error estimte above valid for $\dot u - u = f$?

Exercise Study the case dG0 for $a > 0$.

Exercise (more difficult) Study the cG1 approximation for $\dot u + u = f$ and show that

$$|(u - U)(T)| \leq T \max_{[0,T]}|k^2 \ddot u|.$$

Solution:

Interpolation exercises

With $\bar{w}$ denoting the piecewise constant mean value of w, and $\hat{u}$ the piecewise linear interpolant of u, and further $\theta = \hat{u} - U$, $\rho = u - \hat{u}$ and Φ the cG1-approximation of ϕ such that

$$\int_0^T (-\dot{\Phi} + \Phi)\, v = 0 \quad \text{for all piecewise const. } v,$$

we have

$$|e(T)|^2 = |\theta(T)|^2 = \phi(T)\,\theta(T) + \underbrace{\int_0^T (-\dot{\Phi} + \Phi)\,\hat{\theta}}_{=0}$$

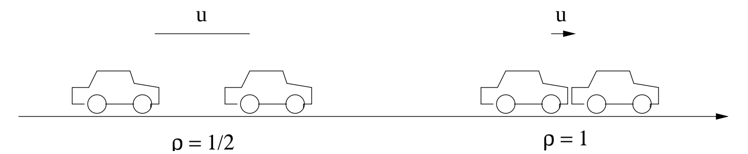
$$= \int_0^T (\dot{\theta} + \theta)\,\bar{\Phi} = -\int_0^T (\dot{\rho} + \rho)\,\bar{\Phi}$$
$$= -\int_0^T \rho\,\bar{\Phi} \le \int_0^T |\bar{\Phi}|\, \max_{[0,T]} |k^2 \ddot{u}| \le T |e(T)| \max_{[0,T]} |k^2 \ddot{u}|.$$

Lecture 11:

Convection problems

A traffic flow problem

Consider traffic flow, with denisity $\rho = \rho(x,t)$ and velocity u, along a highway.



To model the traffic flow we consider a segment $[a,b]$ of the highway.

A traffic flow problem

The differnece of the inflow $u(a,t)\,\rho(a,t)$ of cars at $x = a$ and the outflow $u(b,t)\,\rho(b,t)$ at $x = b$ give rise to an increase of cars in the segment according to

$$\frac{d}{dt}\int_a^b \rho(x,t)\,dx = \int_a^b \dot{\rho}(x,t)\,dx = u(a,t)\,\rho(a,t) - u(b,t)\,\rho(b,t)$$
$$= -\int_a^b (u\,\rho)'\,dx,$$

or

$$\int_a \dot{\rho} + (u\rho)'\,dx = 0.$$

Because a and b are arbitrary, we conclude that

$$\dot{\rho} + (u\rho)' = 0 \quad \text{for } x \in R, t > 0.$$

A traffic flow problem

Assume now that we measure density and velocity in units such that $0 \le \rho, u \le 1$ and adopt the *model* $u = 1 - \rho$ for the velocity of the traffic (why is this a reasonable model?). We then obtain

$$\dot{\rho} + (\rho - \rho^2)' = 0,$$

or, by the chain rule,

$$\dot{\rho} + (1 - 2\rho)\rho' = 0.$$

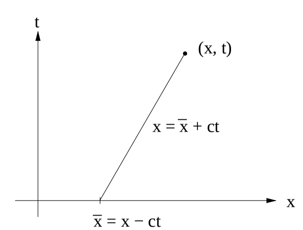
This is a *non-linear* convection equation (treated in the course PDE-nonlinear problems).

A traffic flow problem

Modifying the velocity model to $u = \beta - \epsilon\rho'/\rho$ with $\beta > 0$ and $\epsilon > 0$ (motivate this model!) we obtain a non-linear convection-diffusion equation

$$\dot{\rho} + \beta\rho' - (\epsilon\rho')' = 0,$$

For $\epsilon = 0$ (and β constant) the solution is given by $\rho(x,t) = \rho_0(x - \beta t)$, because ρ is constant in the space-time direction $(\beta, 1)$:



A traffic flow problem

Changing notation $\rho \to u$ we obtain

$$\ddot{u} + \beta u' - (\epsilon u')' = 0.$$

Compare to the Navier-Stokes equations for incompressible fluids:

$$\dot{u} + (\beta \cdot \nabla)u - \epsilon\Delta u + \nabla p = 0, \quad \text{div}\, u = 0,$$

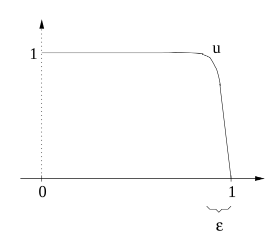
where $\beta = u$, $u = (u_1, u_2, u_3)$ is the velocity, p is the pressure, and $\epsilon = 1/\text{Re}$ where Re is the Reynolds number. These equations are difficult to solve for ϵ small due to boundary layers and turbulence. Typical values of Re are $10^5 - 10^7$, that is $\epsilon << 1$.

A traffic flow problem

Example Consider the model problem

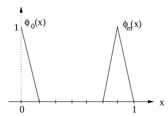
$$u' - \epsilon u'' = 0, 0 < x < 1, u(0) = 1, u(1) = 0,$$

with exact solution $u(x) = c(e^{1/\epsilon} - e^{x/\epsilon})$ and $c = 1/(e^{1/\epsilon} - 1)$. Note the outflow boundary layer of width ϵ near $x = 1$.



A traffic flow problem

With the usual ansatz $U(x) = U_0\phi_0(x) + .. + U_m\phi_m(x)$:



and the node values U_i determined by

$$\int_0^1 \phi_j\, U' + \phi_j'\, \epsilon U' = 0, \quad j = 0, .., m,$$

we obtain the equations

$$\frac{1}{2}(U_{j+1} - U_{j-1}) + \epsilon/h\,(2U_j - U_{j-1} - U_{j+1}) = 0,$$

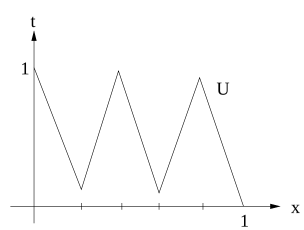
where $U_0 = 1$ and $U_m = 0$. We note that we may rewrite this as

A traffic flow problem

$$\underbrace{\frac{U_{j+1} - U_{j-1}}{2h}}_{\approx u'(x_j)} \quad -\epsilon \underbrace{\frac{U_{j+1} - 2U_j + U_{j-1}}{h^2}}_{\approx u''(x_j)} = 0.$$

central difference

For ϵ very small we thus have $U_{j+1} \approx U_{j-1}$, that is, every second node value is essentially the same, resulting in the following behaviour:



A traffic flow problem

A better method is to approximate $u'(x_j)$ with the "upwind" difference quotient

$$u'(x_j) \approx \frac{U_j - U_{j-1}}{h}, \qquad (1)$$

which gives better stability, while (formally) lesser accuracy. However, accuracy is not relevant without stability!
A more systematic way of stabilizing the finite element method is to use the *streamline diffusion* concept, that is, to use $v + \frac{1}{2}h\beta v'$ (rather that just v) as a "test quantity". For the model problem we obtain

A traffic flow problem

$$\int_0^1 (v + \frac{1}{2}hv')\,(U' - \epsilon U'') = \int_0^1 (v + \frac{1}{2}hv')\, f.$$

Here the term $\int_0^1 vU''$ may be integrated by parts as usual, while the term $\int_0^1 hv'\,U''$ must be interpreted as

sum$_j \displaystyle\int_{I_j} hv'\,U'' \quad (= 0$ in the present case of piecewise linears$)$,

Note also that for $v = \phi_j$ we obtain

A traffic flow problem

$$\int \frac{1}{2} h\, \phi_j'\, U' = U_j - \frac{1}{2} U_{j+1} - \frac{1}{2} U_{j-1},$$

which together with

$$\int \phi_j\, U' = \frac{U_{j+1} - U_{j-1}}{2},$$

gives

$$U_j - U_{j-1},$$

like in the “upwind” scheme.

A traffic flow problem

The method can also be seen as a weighted mixture of the cG1 method with the *least squares* method, where U minimizes $||U' - f||$ if

$$\int v'\, U' = \int v'\, f.$$

Recall that we may write the equation as $Au = f$ with $A = \frac{d}{dx}$ and $A^\top = -\frac{d}{dx}$ as

$$A^\top A u = A^\top f,$$

or

$$-u'' = -f,$$

A traffic flow problem

If we multiply this with v and integrate by parts we obtain

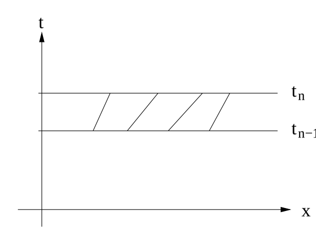
$$\int v'\, u' = \int v'\, f,$$

as above.

A traffic flow problem

For time dependent convection problems we may use oriented space-time meshes

$$\dot{u} + \beta u' - \epsilon u'' = f$$



A traffic flow problem

We make the ansatz $U(x,t)$ such that U is piecewise linear in x and piecewise constant in the directions of $(\beta, 1)$ in space-time. Combining this with the streamline diffusion and artificial viscosity methods we obtain

$$\int_{I_N}\int_\Omega (v+\frac{1}{2}h\beta v')\,(\dot U-\beta U') = \int_{I_n}\int_\Omega (v+\frac{1}{2}h\beta v')\,f.$$

Lecture 12:

Multi grid

Multigrid

We recall that *stiffness matrix* and *load vector* for the problem

$$-u''=f \quad 0<x<1,$$

with homogeneous Dirichlet b.c. takes the form

$$A=\frac{1}{h}\begin{bmatrix} 2 & -1 & & 0 \\ -1 & 2 & -1 & \\ & . & . & . \\ 0 & & -1 & 2 \end{bmatrix} \text{ resp. } b=\begin{bmatrix} b_1 \\ . \\ . \\ b_m \end{bmatrix},$$

where $b_j=\int_{x_{j-1}}^{x_{j+1}}\phi_j\,f \approx hf_j$, $f_j:=f(x_j)$ and the node values

Multigrid, cont.

U_i in the ansatz $U(x)=U_1\,\phi_1(x)+..+U_m\,\phi_m(x)$ is determined by $AU=b$, where $U=\begin{bmatrix} U_1 \\ . \\ . \\ U_m \end{bmatrix}$. The equation in row j is

$$-\frac{1}{h}U_{j-1}+\frac{2}{h}U_j-\frac{1}{h}U_{j+1}=b_j,$$

which can also be written as

$$-\frac{\frac{U_{j+1}-U_j}{h}-\frac{U_j-U_{j-1}}{h}}{h}=\frac{1}{h}b_j\approx f_j,$$

where the left hand side can be interpreted as

Finite differences

a *finite difference* approximation of $-u''(x_j)$. We note that from Taylors formula

$$\begin{array}{rl} & u_{j+1} - u_j = hu'_j + \frac{1}{2}h^2u''_j + \frac{1}{6}h^3u'''_j + O(h^4) \\ + & u_{j-1} - u_j = -hu'_j + \frac{1}{2}h^2u''_j - \frac{1}{6}h^3u'''_j + O(h^4) \\ -- & ---------------------- \\ & u_{j+1} - 2u_j + u_{j-1} = h^2u''_j + O(h^4), \end{array}$$

that is

Finite differences

$$u''(x_j) = \frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} + O(h^2),$$

while before we have concluded that

$$u'(x_j) = \frac{u_{j+1} - u_{j-1}}{2h} + O(h^2),$$

and

$$u'(x_j) = \frac{u_j - u_{j-1}}{h} + O(h).$$

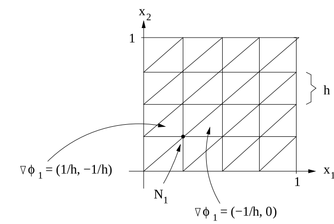
Exercise Find the difference schemes corresponding to dG0cG1 and cG1cG1, respectivly, for the heat equation $\dot{u} - u'' = f$.

Discretization

The stiffness matrix for $\Omega = \square$ &

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

and triangulation as in figure, with $N = m^2$ inner nodes



Discretization

and

$$\begin{array}{l} \int_\Omega \nabla\phi_1 \cdot \nabla\phi_1 = 4, \\ \int_\Omega \nabla\phi_2 \cdot \nabla\phi_1 = -1, \\ \int_\Omega \nabla\phi_{m+1} \cdot \nabla\phi_1 = -1, \end{array}$$

is given by

The discrete problem

$$A = \left[\begin{array}{cccc|ccc|c}
4 & -1 & & 0 & -1 & 0 & & 0 \\
-1 & 4 & . & & 0 & -1 & . & \\
 & . & . & . & & . & . & \\
0 & & -1 & 4 & & & -1 & \\
\hline
-1 & & & & 4 & -1 & & \\
 & . & & & -1 & 4 & . & \\
 & & & & & . & . & \\
0 & & & & & & &
\end{array}\right]$$

Algebraic structure

We note that A is *sparse* (with few non-zero entries), and *banded* with bandwidth $d = 2m + 1$. In fact, here only $\approx 5N$ of totally N^2 matrix entries are non-zero. So, why store all these in the computer (cf. sparse routines in Matlab).
In the course of Gaussian elimination involving a banded matrix one needs $\approx \frac{2}{3}N^3$ operations to solve $AU = b$, or $\approx d^2N$ if A is stored as a banded matrix.
Exercise How many operations does this make if $h = 1/100$?

Iterative solution

We start by considering the *gradient method* or *steepest descent*. We note that $U = \begin{bmatrix} U_1 \\ . \\ . \\ U_m \end{bmatrix}$ minimizes

$$F(U) = \frac{1}{2}U^\top AU - b^\top U.$$

Exercise Show this, using the fact that the corresponding $U(x)$ minimies

$$F(U) = \frac{1}{2}\int_\Omega |\nabla U|^2 - \int_\Omega U\, f.$$

The gradient method

We easily compute the *gradient* g to $F(U)$

$$g = AU - b.$$

The gradientmethod is now: Given a suitable starting vector $U^{(0)}$, compute

$$U^{(j+1)} = U^{(j)} - \alpha_j g(U^{(j)}), \quad \alpha_j > 0,$$

where α_j can be chosen *optimal* or *fixed*.
Exercise Show that α_j optimal for

$$\alpha_j = \frac{(b^\top - U^\top A)g}{g^\top Ag}.$$

The gradient method, cont.

The gradient method "quickly damps high frequency components of the error, but then converges slowly".

Multi (2) grid

Put $v = u - U^{(1)}$ and $u = U^{(1)} + v$ and note that

$$-\Delta v = f + \Delta U^{(1)} =: g.$$

Now compute $V \approx v$ using finite elements of a coarser mesh, which should be okey since v is "smooth". If V is computed with a mesh size $H = 2h$, the number of operations involved will be

$$N + (\frac{N}{4})^2 << N^2 \quad \text{in case of two space dimensions.}$$

Note that $d^2 N \approx N^2$ in 2D, and that a gradient step requires $O(N)$ operations.

Multigrid

If also v is computed using the same principle, we obtain the multigrid method with number of operations

$$N + \frac{N}{4} + \frac{N}{16} + ... + 1 \approx N \quad !!!,$$

that is, the equation can be solved without work. Note that just to write down the solution requires $O(N)$ operations.

Lecture 13:

Nonlinear problems

Non-linear problems

Example

$$-\text{div}(a(u)\,\text{grad}u) = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \qquad (2)$$

with $a(u)$ a temperature dependent heat conductivity. The variational form of the problem is now

$$\int_\Omega \nabla v \cdot a(u)\nabla u = \int_\Omega v\, f \quad \forall v \text{ with } v = 0 \text{ on } \partial\Omega.$$

The finite element method uses the ansatz $U(x) = U_1\,\phi_1 + .. + U_m\,\phi_m$ determined by

The discrete problem

$$\int_\Omega \nabla\phi_j \cdot a(U)\nabla U = \int_\Omega \phi_j\, f, \quad j = 1, .., m,$$

where $U = \begin{bmatrix} U_1 \\ . \\ . \\ U_m \end{bmatrix}$. Note that this is now a *non-linear* system of equations, which cannot be written simply as $AU = b$, with a given matrix A, but well as

$$A(U)\,U = b,$$

where

The discrete problem, cont.

$$A(U) = \begin{bmatrix} \int_\Omega \nabla\phi_1 \cdot a(U)\nabla\phi_1 & \dots & \int_\Omega \nabla\phi_1 \cdot a(U)\nabla\phi_m \\ .. & \dots & .. \\ \int_\Omega \nabla\phi_m \cdot a(U)\nabla\phi_1 & \dots & \int_\Omega \nabla\phi_m \cdot a(U)\nabla\phi_m \end{bmatrix}.$$

Picard iteration

The simplest possible scheme for iterative solution of the non-linear system of equtions for U is

$$A(U^{(j)})\,U^{(j+1)} = b,$$

which gives a sequence $U^{(0)}$, $U^{(1)}$, $U^{(2)}$,.., with $U^{(j)} \to U$ if we are reasonably lucky. Can be expected to converge if A' is small, because

$$A(U^{(j)})(U^{(j+1)} - U) = \underbrace{[A(U) - A(U^{(j)})]}_{\approx -A'\,(U^{(j)}-U)}\,U$$

Picard iteration, cont.

Why not consider the scheme

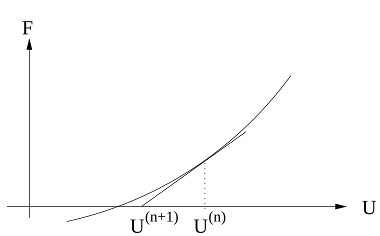
$$A(U^{(j+1)})\,U^{(j)} = b,$$

- non-linear
- unnatural, boundary conditions?
- convergence unlikely

Newtons method

We like to have

$$F_j(U) = \int_\Omega a(U)\nabla\phi_j \cdot \nabla U - \int_\Omega \phi_j\, f = 0, \quad j = 1, .., m.$$



Seek increment $W = U^{(j+1)} - U^{(j)}$ such that

Newton, cont.

$$U^{(j+1)} = U^{(j)} - \frac{F(U^{(j)})}{F'(U^{(j)})},$$

or

$$F'(U^{(j)})W = -F(U^{(j)}),$$

where

Newton, cont.

$$F'(U) = \begin{bmatrix} \frac{\partial F_1}{\partial U_1} & \dots & \frac{\partial F_1}{\partial U_m} \\ \dots & .. & \dots \\ \frac{\partial F_m}{\partial U_1} & \dots & \frac{\partial F_m}{\partial U_m} \end{bmatrix}, \quad F(U) = \begin{bmatrix} F_1(U) \\ .. \\ F_m(U) \end{bmatrix}$$

Newton, cont.

Exercise Investigate the convergece of Picard iteration if $a(u) = 1 + \epsilon u^2$, $\epsilon > 0$ small.
Exercise Compute $\frac{\partial F_1}{\partial U_2}$ for $a(u) = 1 + \epsilon u^2$.
Exercise Solve $-(a(u)u')' = 1$ for $0 < x < 1$, $u(0) = u(1) = 0$ with $a(u) = 1 + \epsilon u^2$ using a) Picard iteration b) Newton
Exercise Seek a solution to $-u'' = f(u)$ for $0 < x < 1$ and b.c. $u(0) = u(1) = 0$ for suitable $f(u)$ of interest.

The new trend - relax !

For the solution of $-\Delta u = f$ in Ω we have

$$\int_\Omega \nabla v \cdot \nabla u - \int_{\partial\Omega} v\, \partial_n u = \int_\Omega v\, f,$$

where $\partial_n = n \cdot \nabla$, and given boundary conditions $u = u_0$ on $\partial\Omega$ can be written as

$$\int_{\partial\Omega} w\, u = \int_{\partial\Omega} w\, u_0 \quad \text{(with } [u] = u - u_0 \text{ as } \int_{\partial\Omega} w\,[u] = 0).$$

Adding these together we obtain

The new trend - relax, cont.

$$\underbrace{\int_\Omega \nabla v \cdot \nabla u - \int_{\partial\Omega} v\, \partial_n u + \int_{\partial\Omega} w\, u}_{a(v,u)} = \int_\Omega v\, f + \int_{\partial\Omega} w\, u_0.$$

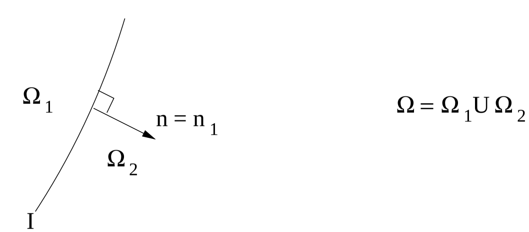
We now want to choose $w = \alpha\, \partial_n v + \beta v$ such that, in particular, the bilinear form $a(v,u)$ becomes positive definite, that is, $a(v,v) > 0$! How choose α and β?
For example, $\alpha = 1$ and $\beta > 0$ implies

$$a(v,v) = \int_\Omega \nabla v \cdot \nabla v + \beta \int_{\partial\Omega} v^2 > 0 \quad \text{for all } v \neq 0.$$

This is okey also for $\alpha \neq 1$ if β is sufficiently large $\approx \frac{\gamma}{h}$ for $v \in V_h$.

The new trend - relax, cont.

For an *interface* I with $u_1 = u_2$ and $\partial_n u_1 = \partial_n u_2$



we have

$$\int_{\Omega_i} \nabla v \cdot \nabla u_i - \int_I v_i \partial_{n_i} u_i = \int_{\Omega_i} v\, f \quad i = 1, 2,$$

Adding these, we obtain with $n = n_1$

The new trend - relax, cont.

$$\int_\Omega \nabla v \cdot \nabla u - \int_I (v_1\,\partial_n u_1 - v_2\,\partial_n u_2) = \int_\Omega v\, f.$$

Here

$$\overbrace{v_1\,\partial_n u_1 - v_2\,\partial_n u_2} = \underbrace{\frac{1}{2}(v_1+v_2)}_{<v>}\ \underbrace{(\partial_n u_1 - \partial_n u_2)}_{[\partial_n u]} + \underbrace{(v_1 - v_2)}_{\substack{[v]\\ \text{jump}}}\ \underbrace{\frac{1}{2}(\partial_n u_1 + \partial_n u_2)}_{\substack{<\partial_n u>\\ \text{mean}}},$$

where $<>$ denotes mean value and $[\,]$ jump. We now add $\int_I [u]\,w + [\partial_n u]\ <v>$, with $w = \alpha\ <\partial_n v> + \beta\,[v]$, to enforce continuity of u and $\partial_n u$ across the interface I. Disregarding boundary conditions, we obtain

The new trend - relax, cont.

$$\underbrace{\int_\Omega \nabla v \cdot \nabla u - \int_I [v]\ <\partial_n u> + \int_I w\,[u]}_{a(v,u)} = \int_\Omega v\, f,$$

where $w = \alpha < \partial_n v > + \beta[v]$. Again $a(v,v) > 0$ for all $v \in V_h$ if $\beta = O(1/h)$ is sufficiently large.
Applying this to all interelement interfaces in a discretization gives *discontinous Galerkin methods* with many possibilities!!