TMA225 Differential Equations and Scientific Computing, part A

Solutions to Problems Week 7

Note: Complete except for the *-problems.

October 15, 2002

Week 7:

Problem 1. Compute ∇u, $n \cdot \nabla u$, and $\triangle u$ for
(a) $u(x, y) = xy$; $\quad n = (1, 0)$,
(b) $u(x, y) = \sin(x)\cos(y)$; $\quad n = (1, 1)$,
(c) $u(x, y) = \log(r)$ where $r = \sqrt{x^2 + y^2}$ $\quad (r \neq 0)$; $\quad n = (x, y)$.
Solution:
(a) $\nabla u = (\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}) = (y, x)$, so $n \cdot \nabla u = y$, and $\triangle u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial y}{\partial x} + \frac{\partial x}{\partial y} = 0$.
(b) $\nabla u = (\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}) = (\cos(x)\cos(y), -\sin(x)\sin(y))$, so
$n \cdot \nabla u = \cos(x)\cos(y) - \sin(x)\sin(y) = \cos(x + y)$, and
$\triangle u = \frac{\partial(\cos(x)\cos(y))}{\partial x} - \frac{\partial(\sin(x)\sin(y))}{\partial y} = -\sin(x)\cos(y) - \sin(x)\cos(y) = -2\sin(x)\cos(y)$.
(c) $\nabla u = (\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}) = (\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}) = \frac{1}{r^2}(x, y)$, so $n \cdot \nabla u = \frac{1}{r^2}(x^2 + y^2) = 1$, and
$\triangle u = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) = 0$. $\square$

Problem 2. Consider the triangulation of $\Omega = [0, 2] \times [0, 1]$ into 3 triangles drawn in Figure 1. (It is the same triangulation as in *Problem 5 (Week 6).*)

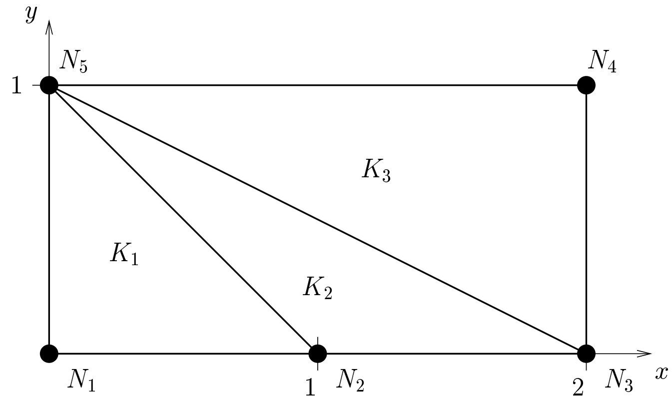

Figure 1: Problem 1 and Problem 4 (Week 7). The triangulation of Ω.

Compute by hand the stiffness matrix A with elements $a_{ij} = \iint_\Omega \nabla\varphi_j \cdot \nabla\varphi_i \, dx\, dy$, $i, j = 1, \ldots, 5$.

Hint: Since $\varphi_i(x, y)$ is linear on each triangle, the gradient $\nabla\varphi_i$ will be a *constant* vector on each triangle. As an example, consider triangle K_1. On this triangle, it is easy to show that $\varphi_1(x, y) = 1 - (x + y)$, $\varphi_2(x, y) = x$, and $\varphi_5(x, y) = y$ (cf. how you did in *Problem 2(a) (Week 5)*). Therefore, on K_1: $\nabla\varphi_1 = (-1, -1)$, $\nabla\varphi_2 = (1, 0)$, and $\nabla\varphi_5 = (0, 1)$. Thus, $a_{11} = \iint_\Omega \nabla\varphi_1 \cdot \nabla\varphi_1 \, dx\, dy = \iint_{K_1} \nabla\varphi_1 \cdot \nabla\varphi_1 \, dx\, dy = \iint_{K_1} 2 \, dx\, dy = 1$. Observe that some matrix elements will get contributions from more than one triangle.
Solution: The matrix A with elements $a_{ij} = \iint \nabla\varphi_j \cdot \nabla\varphi_i \, dxdy$ is clearly symmetric. One easily sees which elements in A that are zero. For example, since φ_1 only is non-zero on triangle K_1 and φ_4 only is non-zero on triangle K_3, we know that $a_{14} = a_{41} = 0$. Similarly we see that $a_{13} = a_{31} = a_{24} = a_{42} = 0$.

Since $\varphi_i(x, y)$ is linear on each triangle, the gradient $\nabla\varphi_i$ will be a constant vector on each triangle.

We now calculate a_{55}. The function φ_5 is non-zero on all triangles. By solving a linear system of equations on each triangle (cf. *Problem 2(a) (Week 5)*), we get that $\varphi_5(x, y) = y$ on triangles K_1 and K_2, and $\varphi_5(x, y) = 1-x/2$ on triangle K_3. So $\nabla\varphi_5 = (0, 1)$ on triangles K_1 and K_2, and $\nabla\varphi_5 = (-1/2, 0)$ on triangle K_3. Thus,

$$\begin{aligned} a_{55} &= \iint_\Omega \nabla\varphi_5 \cdot \nabla\varphi_5 \, dxdy = \iint_{K_1\cup K_2} 1 \, dxdy + \iint_{K_3} 1/4 \, dxdy \\ &= \mu(K_1 \cup K_2) + \frac{1}{4}\mu(K_3) = 1 + 1/4 = 5/4, \end{aligned}$$

where $\mu(K_1 \cup K_2) = 1$ and $\mu(K_3) = 1$ denote the areas of $K_1 \cup K_2$ and K_3 respectively.

We now calculate $a_{12} = a_{21}$. Since $\varphi_1(x, y)$ only is non-zero on triangle K_1 it is enough to integrate over triangle K_1, where $\nabla\varphi_1 = (-1, -1)$ and $\nabla\varphi_2 = (1, 0)$ (see the given Hint in the exercise):

$$\begin{aligned} a_{12} = a_{21} &= \iint_\Omega \nabla\varphi_2 \cdot \nabla\varphi_1 \, dxdy = \iint_{K_1} \nabla\varphi_2 \cdot \nabla\varphi_1 \, dxdy \\ &= \iint_{K_1} -1 \, dxdy = -\mu(K_1) = -1/2. \end{aligned}$$

Similarly we now calculate a_{22}. Since $\varphi_2(x, y)$ only is non-zero on triangles K_1 and K_2 it is enough to integrate over these triangles. On K_1, $\varphi_2(x, y) = x$ so there $\nabla\varphi_2 = (1, 0)$, and on K_2, $\varphi_2(x, y) = 2 - (x + 2y)$ so there $\nabla\varphi_2 = (-1, -2)$. This gives that

$$\begin{aligned} a_{22} &= \iint_{K_1} \nabla\varphi_2 \cdot \nabla\varphi_2 \, dxdy + \iint_{K_2} \nabla\varphi_2 \cdot \nabla\varphi_2 \, dxdy \\ &= \iint_{K_1} 1 \, dxdy + \iint_{K_2} 5 \, dxdy = 1/2 + 5/2 = 3. \end{aligned}$$

In the same way as above one gets that $a_{11} = 1$, $a_{33} = 2$, $a_{44} = \frac{5}{4}$, $a_{43} = a_{34} = -1$, $a_{15} = a_{51} = -\frac{1}{2}$, $a_{25} = a_{52} = -1$, $a_{23} = a_{32} = -\frac{3}{2}$, $a_{35} = a_{53} = \frac{1}{2}$ and $a_{45} = a_{54} = -\frac{1}{4}$. Thus:

$$A = \begin{bmatrix} 1 & -1/2 & 0 & 0 & -1/2 \\ -1/2 & 3 & -3/2 & 0 & -1 \\ 0 & -3/2 & 2 & -1 & 1/2 \\ 0 & 0 & -1 & 5/4 & -1/4 \\ -1/2 & -1 & 1/2 & -1/4 & 5/4 \end{bmatrix}.$$

□

Problem 3. Let $\mathcal{P}(K) = \{v(x) = c_0 + c_1x_1 + c_2x_2,\ c_i \in \mathbf{R},\ i = 1, 2, 3;\ x = (x_1, x_2) \in K\}$ be the space of linear polynomials defined on a triangle K with corners a^1, a^2, and a^3. Derive explicit expressions (in terms of the corner coordinates $a^1 = (a^1_1, a^1_2)$, $a^2 = (a^2_1, a^2_2)$, and $a^3 = (a_1^3,\, a_2^3))$ for the gradients $\nabla\lambda_1$, $\nabla\lambda_2$, $\nabla\lambda_3$ of the basis functions $\lambda_1, \lambda_2, \lambda_3 \in \mathcal{P}(K)$ defined by

$$\lambda_i(a^j) = \begin{cases} 1 & i = j, \\ 0 & i \neq j, \end{cases}$$

with $i, j = 1, 2, 3$. Compare with the corresponding expressions in your Matlab-function `MyFirst2DPoissonAssembler`.

Hint: Use the result from *Problem 3 (Week 6)*.

Solution: Since $v(x) = c_0 + c_1x_1 + c_2x_2$, we have $\nabla v(x) = (c_1, c_2)$. All we have to do to determine the gradients is then to identify the coefficients c_1 and c_2. From *Problem 3 (Week 6)*, we get

$$\begin{aligned}
\nabla\lambda_1 &= \frac{1}{\det A}(a_2^2 - a_2^3,\, a_1^3 - a_1^2), \\
\nabla\lambda_2 &= \frac{1}{\det A}(a_2^3 - a_2^1,\, a_1^1 - a_1^3), \\
\nabla\lambda_2 &= \frac{1}{\det A}(a_2^1 - a_2^2,\, a_1^2 - a_1^1),
\end{aligned}$$

where $\det A = a_1^3a_2^1 + a_1^2a_2^3 - a_1^2a_2^1 - a_1^3a_2^2 - a_1^1a_2^3 + a_1^1a_2^2 = 2\,\mu(K)$, where $\mu(K)$ is the area of K. (Cf. *Problem 4 (Week 6)*.) Note that the gradients are constant, which is a property of a plane. $\square$

Problem 4. Consider once more the triangulation of $\Omega = [0,\, 2] \times [0,\, 1]$ into 3 triangles drawn in Figure 1. Let $\Gamma = \partial\Omega$ denote the boundary of Ω. Assuming that $\gamma(x,\, y) = 1$, $g_D(x,\, y) = 1 + x + y$, and $g_N(x,\, y) = 0$, compute by hand:
(a) The "boundary matrix" R with elements $r_{ij} = \int_\Gamma \gamma\, \varphi_j\, \varphi_i\, ds$, $i, j = 1, \ldots, 5$.
(b) The "boundary vector" rv with elements $\mathrm{rv}_i = \int_\Gamma (\gamma g_D - g_N)\, \varphi_i\, ds$, $i = 1, \ldots, 5$.

Hint: You can either compute the curve integrals analytically or use *Simpson's* formula which is exact in this case.

Solution: Start by dividing and numbering the boundary Γ into five segments Γ_i, $i = 1, \ldots, 5$ according to Figure 2.
(a) The first row of the "boundary matrix" is now computed using Simpson's rule. When doing this we have to keep track of where the basis functions are non-zero. For instance, φ_1 is identically equal to zero on the boundary except on the segments Γ_1 and Γ_5. Further, the value of φ_1 at the midpoints of Γ_1 and Γ_5 is $\frac{1}{2}$. Since $\gamma = 1$ this gives:

$$\begin{aligned}
r_{11} &= \int_\Gamma \varphi_1^2\, ds = \int_{\Gamma_5} \varphi_1^2\, ds + \int_{\Gamma_1} \varphi_1^2\, ds = \int_0^1 \varphi_1(0,\, y)^2\, dy + \int_0^1 \varphi_1(x,\, 0)^2\, dx \\
&= \{\text{Simpson's rule}\} = \frac{1\cdot 1 + 4\cdot\frac{1}{2}\cdot\frac{1}{2} + 0\cdot 0}{6}\cdot 1 + \frac{1\cdot 1 + 4\cdot\frac{1}{2}\cdot\frac{1}{2} + 0\cdot 0}{6}\cdot 1 = \frac{2}{3}.
\end{aligned}$$

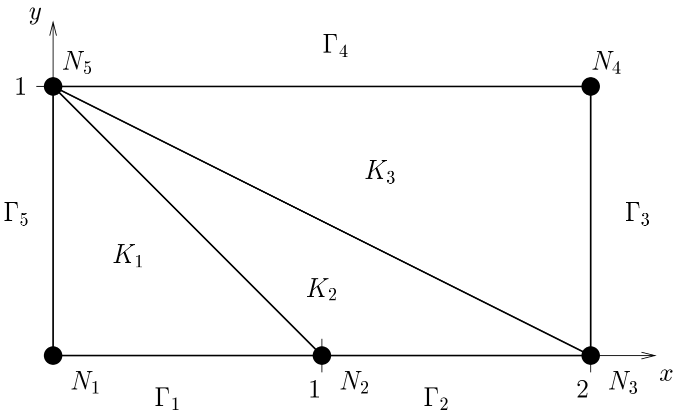

Figure 2: Problem 4 (Week 7). The five segments of Γ.

Since φ_1 and φ_2 are non-zero simultaneously only on Γ_1 we get:

$$r_{12} = \int_{\Gamma_1} \varphi_2\varphi_1 \, ds = \int_0^1 \varphi_2(x,\, 0)\varphi_1(x,\, 0)\, dx = \frac{0\cdot 1 + 4\cdot \frac{1}{2}\cdot\frac{1}{2} + 1\cdot 0}{6}\cdot 1 = \frac{1}{6},$$

and analogously:

$$r_{15} = \int_{\Gamma_5} \varphi_5\varphi_1 \, ds = \int_0^1 \varphi_5(0,\, y)\varphi_1(0,\, y)\, dy = \frac{0\cdot 1 + 4\cdot \frac{1}{2}\cdot\frac{1}{2} + 1\cdot 0}{6}\cdot 1 = \frac{1}{6}.$$

The matrix element $r_{13} = 0$ since φ_1 and φ_3 don't overlap on any boundary segment. The same reasoning leads to $r_{14} = 0$. Similar computations give the rest of the matrix elements (also note that R is symmetric), we just have to remember how long the boundary segments are. (Be careful with the integrals over Γ_4; don't forget that this segment has length 2!) The final result is:

$$R = \begin{bmatrix} \frac{2}{3} & \frac{1}{6} & 0 & 0 & \frac{1}{6} \\ & \frac{2}{3} & \frac{1}{6} & 0 & 0 \\ & & \frac{2}{3} & \frac{1}{6} & 0 \\ & \text{symm.} & & 1 & \frac{1}{3} \\ & & & & 1 \end{bmatrix}.$$

(b) We start by computing the first component rv_1 of the "boundary vector" rv. Since $\gamma = 1$, $g_D = 1 + x + y$ and $g_N = 0$ the integrand becomes $(1 + x + y)\,\varphi_1(x,\, y)$. Note that since φ_1 is non-zero only on Γ_1 and Γ_5 we need only integrate over these two boundary segments. Further note that $y = 0$ on Γ_1 and that $x = 0$ on Γ_5:

$$\mathrm{rv}_1 = \int_\Gamma (1 + x + y)\,\varphi_1(x,\, y)\, ds = \int_{\Gamma_1} (1 + x + 0)\,\varphi_1(x,\, 0)\, ds + \int_{\Gamma_5} (1 + 0 + y)\,\varphi_1(0,\, y)\, ds$$

$$
\begin{aligned}
&=\{ds = dx \text{ on } \Gamma_1;\ ds = dy \text{ on } \Gamma_5\} = \int_0^1 (1+x)\,\varphi_1(x,\,0)\,dx + \int_0^1 (1+y)\,\varphi_1(0,\,y)\,dy \\
&=\{\text{Simpson's rule}\} = \frac{1\cdot 1 + 4\cdot\frac{3}{2}\cdot\frac{1}{2} + 2\cdot 0}{6}\cdot 1 + \frac{1\cdot 1 + 4\cdot\frac{3}{2}\cdot\frac{1}{2} + 2\cdot 0}{6}\cdot 1 \\
&=\frac{4}{6} + \frac{4}{6} = \frac{4}{3}.
\end{aligned}
$$

Continuing in the same way gives the rest of the elements:

$$
\begin{aligned}
\mathrm{rv}_2 &= \int_\Gamma (1+x+y)\,\varphi_2(x,\,y)\,ds = \int_{\Gamma_1} (1+x+0)\,\varphi_2(x,\,0)\,ds + \int_{\Gamma_2} (1+x+0)\,\varphi_2(x,\,0)\,ds \\
&=\{ds = dx \text{ on } \Gamma_1 \cup \Gamma_2\} = \int_0^1 (1+x)\,\varphi_2(x,\,0)\,dx + \int_1^2 (1+x)\,\varphi_2(x,\,0)\,dx \\
&=\{\text{Simpson's rule}\} = \frac{1\cdot 0 + 4\cdot\frac{3}{2}\cdot\frac{1}{2} + 2\cdot 1}{6}\cdot 1 + \frac{2\cdot 1 + 4\cdot\frac{5}{2}\cdot\frac{1}{2} + 3\cdot 0}{6}\cdot 1 \\
&=\frac{5}{6} + \frac{7}{6} = 2,
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{rv}_3 &= \int_\Gamma (1+x+y)\,\varphi_3(x,\,y)\,ds = \int_{\Gamma_2} (1+x+0)\,\varphi_3(x,\,0)\,ds + \int_{\Gamma_3} (1+2+y)\,\varphi_3(2,\,y)\,ds \\
&=\{ds = dx \text{ on } \Gamma_2;\ ds = dy \text{ on } \Gamma_3\} = \int_1^2 (1+x)\,\varphi_3(x,\,0)\,dx + \int_0^1 (3+y)\,\varphi_3(2,\,y)\,dy \\
&=\{\text{Simpson's rule}\} = \frac{2\cdot 0 + 4\cdot\frac{5}{2}\cdot\frac{1}{2} + 3\cdot 1}{6}\cdot 1 + \frac{3\cdot 1 + 4\cdot\frac{7}{2}\cdot\frac{1}{2} + 4\cdot 0}{6}\cdot 1 \\
&=\frac{8}{6} + \frac{10}{6} = 3,
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{rv}_4 &= \int_\Gamma (1+x+y)\,\varphi_4(x,\,y)\,ds = \int_{\Gamma_3} (1+2+y)\,\varphi_4(2,\,y)\,ds + \int_{\Gamma_4} (1+x+1)\,\varphi_4(x,\,1)\,ds \\
&=\{ds = dy \text{ on } \Gamma_3;\ ds = dx \text{ on } \Gamma_4\} = \int_0^1 (3+y)\,\varphi_4(2,\,y)\,dy + \int_0^2 (2+x)\,\varphi_4(x,\,1)\,dx \\
&=\{\text{Simpson's rule}\} = \frac{3\cdot 0 + 4\cdot\frac{7}{2}\cdot\frac{1}{2} + 4\cdot 1}{6}\cdot 1 + \frac{2\cdot 0 + 4\cdot 3\cdot\frac{1}{2} + 4\cdot 1}{6}\cdot 2 \\
&=\frac{11}{6} + \frac{20}{6} = 5\frac{1}{6},
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{rv}_5 &= \int_\Gamma (1+x+y)\,\varphi_5(x,\,y)\,ds = \int_{\Gamma_4} (1+x+1)\,\varphi_5(x,\,1)\,ds + \int_{\Gamma_5} (1+0+y)\,\varphi_5(0,\,y)\,ds \\
&=\{ds = dx \text{ on } \Gamma_4;\ ds = dy \text{ on } \Gamma_5\} = \int_0^2 (2+x)\,\varphi_5(x,\,1)\,dx + \int_0^1 (1+y)\,\varphi_5(0,\,y)\,dy
\end{aligned}
$$

$$=\{\text{Simpson's rule}\} = \frac{2\cdot 1 + 4\cdot 3\cdot \frac{1}{2} + 4\cdot 0}{6}\cdot 2 + \frac{1\cdot 0 + 4\cdot \frac{3}{2}\cdot \frac{1}{2} + 2\cdot 1}{6}\cdot 1$$
$$=\frac{16}{6} + \frac{5}{6} = 3\frac{1}{2}.$$

□

Problem 5. Show that the equation:

$$\iint_\Omega \nabla U \cdot \nabla v \, dx\, dy = \iint_\Omega f v \, dx\, dy \quad \text{for all } v \in V_{h0}, \tag{1}$$

is equivalent to

$$\iint_\Omega \nabla U \cdot \nabla \varphi_i \, dx\, dy = \iint_\Omega f \varphi_i \, dx\, dy \quad \text{for } i = 1, \ldots, N, \tag{2}$$

where N is the number of internal nodes ("*nintnodes*") and $\{\varphi_i\}_{i=1}^N$ is the basis of "tent-functions" in V_{h0}.

Solution:

$\Rightarrow$: We assume that (1) is true:

$$\iint_\Omega \nabla U \cdot \nabla v \, dxdy = \iint_\Omega f v \, dxdy \quad \forall\, v \in V_{h0},$$

and want to show that this implies that (2) is true:

$$\iint_\Omega \nabla U \cdot \nabla \varphi_i \, dxdy = \iint_\Omega f \varphi_i \, dxdy \quad i = 1, \ldots, N.$$

But since (1) holds for all $v \in V_{h0}$ and $\varphi_i \in V_{h0}$, $i = 1, \ldots, N$, it´s for sure that (1) implies (2).

$\Leftarrow$: We now assume that (2) is true:

$$\iint_\Omega \nabla U \cdot \nabla \varphi_i \, dxdy = \iint_\Omega f \varphi_i \, dxdy \quad i = 1, \ldots, N,$$

and want to show that this implies that (1) is true:

$$\iint_\Omega \nabla U \cdot \nabla v \, dxdy = \iint_\Omega f v \, dxdy \quad \forall\, v \in V_{h0}.$$

Since $\{\varphi_i\}_{i=1}^N$ is a basis for V_{h0}, every $v \in V_{h0}$ can be written:

$$v(x,\, y) = \sum_{i=1}^{N} c_i\, \varphi_i(x,\, y)$$

for some constants c_i, $i = 1, \dots, N$. Multiply (2) with arbitrary constants c_i for all $i = 1, \dots, N$. If we add it all together, we get:

$$\iint_\Omega \nabla U \cdot \nabla(c_1\varphi_1)\, dxdy + \iint_\Omega \nabla U \cdot \nabla(c_2\varphi_2)\, dxdy + \cdots + \iint_\Omega \nabla U \cdot \nabla(c_N\varphi_N)\, dxdy$$

$$= \iint_\Omega f c_1\varphi_1\, dxdy + \iint_\Omega f c_2\varphi_2\, dxdy + \cdots + \iint_\Omega f c_N\varphi_N\, dxdy,$$

and hence:

$$\iint_\Omega \nabla U \cdot \nabla(\sum_{i=1}^{N} c_i\varphi_i)\, dxdy = \iint_\Omega f(\sum_{i=1}^{N} c_i\varphi_i)\, dxdy, \tag{3}$$

for arbitrary constants c_i, $i = 1, \dots, N$. Since (3) holds for *every* set of constants $\{c_i\}_{i=1}^N$, we conclude that (1) holds for all $v \in V_{h0}$. $\square$

Problem 6*. Show that the problem: find $U \in V_{h0}$ such that

$$\iint_\Omega \nabla U \cdot \nabla w dx\, dy = \iint_\Omega f w\, dx\, dy \quad \text{for all } w \in V_{h0}, \tag{4}$$

is equivalent to the minimization problem: find $U \in V_{h0}$ such that

$$\frac{1}{2}\iint_\Omega \nabla U \cdot \nabla U\, dx\, dy - \iint_\Omega fU\, dx\, dy = \min_{v \in V_{h0}} \frac{1}{2}\iint_\Omega \nabla v \cdot \nabla v\, dx\, dy - \iint_\Omega f v\, dx\, dy. \tag{5}$$

Solution:

$\square$

Problem 7*.
(a) Consider the quadratic equation

$$at^2 + bt + c = 0, \tag{6}$$

Investigate under what condition on the coefficients a, b, c equation (6) does *not* have two distinct real roots.
(b) Prove the Cauchy-Schwarz inequality:

$$|\iint_\Omega vw\, dx\, dy| \le \|v\|_{L^2(\Omega)}\|w\|_{L^2(\Omega)} \tag{7}$$

Hint: start from the fact that $\|v+tw\|_{L^2(\Omega)}^2 \ge 0$. Expanding $\|v+tw\|_{L^2(\Omega)}^2$ gives a quadratic polynomial which can not have two distinct real roots (why?). Use (a) to prove the Cauchy-Schwarz inequality.
Solution:

$\square$

Problem 8. Calculate $\|\nabla f\|_{L^2(\Omega)}$ where $\Omega = [0,1] \times [0,1]$ and
(a) $f = x_1 x_2^2$.
(b) $f = \sin(nx_1)\sin(mx_2)$ with n and m arbitrary integers. What happens when n, m tends to infinity?
Solution:
(a) Recall that the gradient, ∇f, of a scalar function $f(x_1, x_2)$ is a vector with the partial derivatives of f as components:

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}\right) = (x_2^2,\ 2x_1x_2).$$

Now, using the definition of the L^2-norm we get:

$$\begin{aligned}\|\nabla f\|^2_{L^2(\Omega)} = \iint_\Omega |\nabla f|^2\, dx_1dx_2 &= \iint_\Omega \nabla f \cdot \nabla f\, dx_1dx_2 = \iint_\Omega x_2^4 + 4x_1^2x_2^2\, dx_1dx_2 \\ &= \int_0^1\int_0^1 (x_2^4 + 4x_1^2x_2^2)\, dx_2dx_1 = \int_0^1 \left[\tfrac{1}{5}x_2^5 + \tfrac{4}{3}x_1^2x_2^3\right]_0^1 dx_1 \\ &= \int_0^1 \left(\tfrac{1}{5} + \tfrac{4}{3}x_1^2\right) dx_1 = \left[\tfrac{1}{5}x_1 + \tfrac{4}{9}x_1^3\right]_0^1 = \frac{1}{5} + \frac{4}{9} = \frac{29}{45},\end{aligned}$$

i.e. we have the answer $\|\nabla f\|_{L^2(\Omega)} = \sqrt{\frac{29}{45}}$.
(b) First note that if $n = 0$ and/or $m = 0$ we have $f \equiv 0$ and therefore $\nabla f = (0, 0)$ and $\|\nabla f\|_{L^2(\Omega)} = 0$. If $n \neq 0$ and $m \neq 0$ we first, as in (a), compute the gradient vector ∇f of the function $f(x_1, x_2)$:

$$\nabla f = (n\cos(nx_1)\sin(mx_2),\ m\sin(nx_1)\cos(mx_2)).$$

As above, we then compute:

$$\|\nabla f\|^2_{L^2(\Omega)} = \iint_\Omega \left(n^2\cos^2(nx_1)\sin^2(mx_2) + m^2\sin^2(nx_1)\cos^2(mx_2)\right) dx_1dx_2.$$

This looks a bit nasty but using the well known trigonometric formulas

$$\sin^2\alpha = \tfrac{1}{2}(1 - \cos 2\alpha), \quad \cos^2\alpha = \tfrac{1}{2}(1 + \cos 2\alpha)$$

we can rewrite the integral, which then equals:

$$\iint_\Omega \left(\frac{n^2}{4}(1 + \cos 2nx_1)(1 - \cos 2mx_2) + \frac{m^2}{4}(1 - \cos 2nx_1)(1 + \cos 2mx_2)\right) dx_1dx_2.$$

Since the factors in the integrand are independent of each other we can break the integral into four separate and simple parts. We thus have:

$$\begin{aligned}
\|\nabla f\|^2_{L^2(\Omega)} &= \frac{n^2}{4}\int_0^1 (1+\cos 2nx_1)\,dx_1 \int_0^1 (1-\cos 2mx_2)\,dx_2 \\
&+ \frac{m^2}{4}\int_0^1 (1-\cos 2nx_1)\,dx_1 \int_0^1 (1+\cos 2mx_2)\,dx_2 \\
&= \frac{n^2}{4}\left[x_1 + \frac{\sin 2nx_1}{2n}\right]_0^1 \left[x_2 - \frac{\sin 2mx_1}{2m}\right]_0^1 + \frac{m^2}{4}\left[x_1 - \frac{\sin 2nx_1}{2n}\right]_0^1 \left[x_2 + \frac{\sin 2mx_2}{2m}\right]_0^1 \\
&= \frac{n^2}{4}\left(1+\frac{\sin 2n}{2n}\right)\left(1-\frac{\sin 2m}{2m}\right) + \frac{m^2}{4}\left(1-\frac{\sin 2n}{2n}\right)\left(1+\frac{\sin 2m}{2m}\right).
\end{aligned}$$

If we let n, m tend to infinity the terms involving *sine* tend to zero because of the big terms in the denominators and we are left with

$$\|\nabla f\|^2_{L^2(\Omega)} \sim \frac{n^2}{4} + \frac{m^2}{4} \to \infty \quad n, m \to \infty.$$

This can be understood if we consider the effect of n in the expression $\sin nx_1$. The integer n determines how fast the function will oscillate, i.e., the frequency. As n tends to infinity the function will oscillate increasingly faster, causing its derivative to become large. And since the norm is a measure of the gradient's size, it will become infinite in the limit. □

Problem 9. Let $u = x_1x_2^2$ and $a = 1 + x_2^2$. Calculate
(a) ∇u.
(b) Δu.
(c) $\nabla \cdot a\nabla u$.
Solution:
(a)

$$\nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}\right) = (x_2^2,\, 2x_1x_2)$$

(b)

$$\Delta u = \nabla \cdot \nabla u = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right) \cdot (x_2^2,\, 2x_1x_2) = 0 + 2x_1 = 2x_1$$

(c)

$$\begin{aligned}
\nabla \cdot (a\nabla u) &= \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right) \cdot (1+x_2^2)(x_2^2,\, 2x_1x_2) \\
&= \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right) \cdot (x_2^2 + x_2^4,\, 2x_1x_2 + 2x_1x_2^3)
\end{aligned}$$

$$= \quad 2x_1 + 6x_1x_2^2$$

□

Problem 10. Consider the problem: find u such that

$$
\begin{aligned}
-\Delta u + cu &= f && \text{in } \Omega, && (8)\\
u &= g_D && \text{on } \Gamma_D, && (9)\\
-n\cdot\nabla u &= g_N && \text{on } \Gamma_N, && (10)
\end{aligned}
$$

where $c = c(x,\, y) \geq 0$, with the usual notation.
(a) Derive a finite element method for this problem using approximation of the Dirichlet boundary condition.
(b) Prove that the finite element solution is unique when 1. $c > 0$ and 2. Γ_D is non-empty.
Solution:
(a) We approximate the Dirichlet boundary condition (9) by

$$-n\cdot\nabla u = \gamma(u - g_D) \quad \text{on } \Gamma_D, \tag{11}$$

where $\gamma >> 0$.
Multiply the differential equation (8) by a function $v = v(x,\, y)$ and integrate over Ω:

$$-\iint_\Omega (\Delta u)v\,dxdy + \iint_\Omega cuv\,dxdy = \iint_\Omega fv\,dxdy.$$

Integrate by parts in the first term:

$$-\int_\Gamma (n\cdot\nabla u)v\,ds + \iint_\Omega \nabla u\cdot\nabla v\,dxdy + \iint_\Omega cuv\,dxdy = \iint_\Omega fv\,dxdy.$$

Use the boundary conditions (10) and (11) to replace $-(n\cdot\nabla u)$ in the boundary integral:

$$\int_{\Gamma_N} g_N v\,ds + \int_{\Gamma_D} \gamma(u-g_D)v\,ds + \iint_\Omega \nabla u\cdot\nabla v\,dxdy + \iint_\Omega cuv\,dxdy = \iint_\Omega fv\,dxdy.$$

We now state the *variational formulation*: Find $u \in V$ such that

$$
\begin{aligned}
&\int_{\Gamma_D} \gamma uv\,ds + \iint_\Omega \nabla u\cdot\nabla v\,dxdy + \iint_\Omega cuv\,dxdy = \\
&\int_{\Gamma_D} \gamma g_D v\,ds - \int_{\Gamma_N} g_N v\,ds + \iint_\Omega fv\,dxdy \quad \text{for all } v\in V,
\end{aligned} \tag{12}
$$

where V is the space of functions that are smooth enough for the integrals in (12) to exist.

The corresponding *Finite Element Method* reads: Find $U \in V_h$ such that

$$\int_{\Gamma_D} \gamma Uv\,ds + \iint_\Omega \nabla U\cdot\nabla v\,dxdy + \iint_\Omega cUv\,dxdy =$$

$$\int_{\Gamma_D} \gamma g_D v \, ds - \int_{\Gamma_N} g_N v \, ds + \iint_\Omega f v \, dxdy \quad \text{for all } v \in V_h, \tag{13}$$

where V_h is the space of continuous, piece-wise linear functions on a given triangulation of Ω.
(b) Assume that there are *two* solutions U_1, $U_2 \in V_h$ to (13):

$$\int_{\Gamma_D} \gamma U_1 v \, ds + \iint_\Omega \nabla U_1 \cdot \nabla v \, dxdy + \iint_\Omega c U_1 v \, dxdy =$$
$$\int_{\Gamma_D} \gamma g_D v \, ds - \int_{\Gamma_N} g_N v \, ds + \iint_\Omega f v \, dxdy \quad \text{for all } v \in V_h,$$

$$\int_{\Gamma_D} \gamma U_2 v \, ds + \iint_\Omega \nabla U_2 \cdot \nabla v \, dxdy + \iint_\Omega c U_2 v \, dxdy =$$
$$\int_{\Gamma_D} \gamma g_D v \, ds - \int_{\Gamma_N} g_N v \, ds + \iint_\Omega f v \, dxdy \quad \text{for all } v \in V_h.$$

Subtraction gives:

$$\int_{\Gamma_D} \gamma (U_1 - U_2) v \, ds + \iint_\Omega \nabla (U_1 - U_2) \cdot \nabla v \, dxdy + \iint_\Omega c (U_1 - U_2) v \, dxdy = 0,$$

for all $v \in V_h$. Now choose $v = U_1 - U_2 \in V_h$:

$$\int_{\Gamma_D} \gamma (U_1 - U_2)^2 \, ds + \iint_\Omega |\nabla (U_1 - U_2)|^2 \, dxdy + \iint_\Omega c (U_1 - U_2)^2 \, dxdy = 0. \tag{14}$$

Since all three terms on the left-hand side are non-negative they must all be equal to 0:

$$\int_{\Gamma_D} \gamma (U_1 - U_2)^2 \, ds = 0, \tag{15}$$

$$\iint_\Omega |\nabla (U_1 - U_2)|^2 \, dxdy = 0, \tag{16}$$

$$\iint_\Omega c (U_1 - U_2)^2 \, dxdy = 0. \tag{17}$$

We now consider the two cases separately:
1. If $c > 0$ equation (17) immediately implies that $U_1 - U_2 = 0$ in Ω, i.e., $U_1 = U_2$ in Ω.
2. If we only know that $c \geq 0$, but Γ_D is non-empty, we can first use (16) to conclude that $|\nabla (U_1 - U_2)| = 0$ in Ω, i.e., $U_1 - U_2$ is *constant* in Ω. Then we use (15) to conclude that $U_1 - U_2 = 0$ on Γ_D, but then the constant must be 0 and we have that $U_1 - U_2 = 0$ in Ω, i.e., $U_1 = U_2$ in Ω.

Remark. Since existence and uniqueness is equivalent for quadratic linear systems of equations, we have also proved existence of a solution to our Finite Element Method.

$\square$

Problem 11. Let K be a triangle with corners $(0,0)$, $(0,1)$, and $(1,0)$, and let $f(x_1, x_2) = x_1^2 + x_2$. Calculate

$$\iint_K f(x_1, x_2)\, dx_1 dx_2,$$

using
(a) one-point ("center of mass") quadrature,
(b) corner ("node") quadrature,
(c) mid-point (of the triangle sides) quadrature.
Also compute the integral analytically and compare with your results above.
Solution: Denote the area of K by $\mu(K)$, i.e., $\mu(K) = \frac{1}{2}$.
(a) The co-ordinates for the *center of mass* of a triangle, (x_{CM}, y_{CM}), are the mean values of the co-ordinates of the corners:

$$(x_{CM}, y_{CM}) = \frac{(0,\,0) + (0,\,1) + (1,\,0)}{3} = (\frac{1}{3}, \frac{1}{3}).$$

Thus:

$$\iint_K f(x_1, x_2)\, dx_1 dx_2 \approx f(x_{CM}, y_{CM})\, \mu(K) = \left(\left(\frac{1}{3}\right)^2 + \frac{1}{3}\right) \cdot \frac{1}{2} = \frac{2}{9}.$$

(b)

$$\iint_K f(x_1, x_2)\, dx_1 dx_2 \approx \frac{f(0,\,0) + f(0,\,1) + f(1,\,0)}{3}\, \mu(K) = \frac{0+1+1}{3} \cdot \frac{1}{2} = \frac{1}{3}.$$

(c)

$$\iint_K f(x_1, x_2)\, dx_1 dx_2 \approx \frac{f(0,\,1/2) + f(1/2,\,0) + f(1/2,\,1/2)}{3}\, \mu(K)$$
$$= \frac{1/2 + 1/4 + 3/4}{3} \cdot \frac{1}{2} = \frac{1}{4}.$$

We know that the quadrature rule in (c) should give the exact result in this case, since f is a polynomial of degree 2. We check:

$$\iint_K f(x_1, x_2)\, dx_1 dx_2 = \int_0^1 \left(\int_0^{1-x_1} (x_1^2 + x_2)\, dx_2 \right) dx_1$$
$$= \int_0^1 \left[x_1^2 x_2 + \frac{1}{2} x_2^2 \right]_{x_2=0}^{x_2=1-x_1} dx_1 = \int_0^1 \left(x_1^2(1-x_1) + \frac{1}{2}(1-x_1)^2 \right) dx_1$$

$$= \int_0^1 \left(\frac{1}{2} - x_1 + \frac{3}{2}x_1^2 - x_1^3\right) dx_1 = \left[\frac{1}{2}x_1 - \frac{1}{2}x_1^2 + \frac{1}{2}x_1^3 - \frac{1}{4}x_1^4\right]_{x_1=0}^{x_1=1} = \frac{1}{2} - \frac{1}{2} + \frac{1}{2} - \frac{1}{4} = \frac{1}{4}!$$

$\square$

Problem 12. Let K be a triangle with corners $(0,0)$, $(0,1)$, and $(1,0)$.
(a) Calculate the three basis functions $\lambda_i,\ i = 1,2,3$, for the space $\mathcal{P}(K)$ of linear functions defined on K.
(b) Calculate the 3×3 element mass matrix with elements $m_{ij} = \iint_K \lambda_j \lambda_i \, dx\, dy$ approximately using corner quadrature.
(c) Calculate the 3×3 element stiffness matrix with elements $a_{ij} = \iint_K \nabla\lambda_j \cdot \nabla\lambda_i \, dx\, dy$.
Solution: Denote the area of K by $\mu(K)$, i.e., $\mu(K) = \frac{1}{2}$. We also introduce the node numbering $N_1 = (0,0)$, $N_2 = (0,1)$, and $N_3 = (1,0)$.
(a) You can compute the basis functions in the same way as you did in *Problem 2(a) (Week 5)*. An alternative is to argue as follows: The basis function $\lambda_3(x,\,y)$ is equal to 1 in $(1,0)$ and is equal to 0 for $x = 0$. It therefore has to be $\lambda_3(x,\,y) = x$, since this is a linear function that obviously satisfies these two requirements. (And linear functions are uniquely determined by their nodal values.) By the same argument we have $\lambda_2(x,\,y) = y$. Finally we know that $\lambda_1(x,\,y) + \lambda_2(x,\,y) + \lambda_3(x,\,y) \equiv 1$ since the sum is a linear function that is equal to 1 in all three nodes. Therefore $\lambda_1(x,\,y) = 1 - \lambda_3(x,\,y) - \lambda_2(x,\,y) = 1 - x - y$.
(b) With corner (node) quadrature we approximate:

$$m_{ij} = \iint_K \lambda_j \lambda_i \, dx\, dy \approx \frac{\lambda_j(0,\,0)\lambda_i(0,\,0) + \lambda_j(0,\,1)\lambda_i(0,\,1) + \lambda_j(1,\,0)\lambda_i(1,\,0)}{3}\,\mu(K).$$

If $i \neq j$ at least one of the factors λ_j and λ_i is zero in each corner and therefore $m_{ij} = 0$. If $i = j$ we get:

$$\begin{aligned} m_{ii} &= \iint_K \lambda_i \lambda_i \, dx\, dy \approx \frac{\lambda_i(0,\,0)\lambda_i(0,\,0) + \lambda_i(0,\,1)\lambda_i(0,\,1) + \lambda_i(1,\,0)\lambda_i(1,\,0)}{3}\,\mu(K) \\ &= \frac{1^2}{3} \cdot \frac{1}{2} = \frac{1}{6}, \end{aligned}$$

since λ_i is equal to 1 in one node and equal to 0 in the other two nodes.

The final result is therefore:

$$\begin{bmatrix} 1/6 & 0 & 0 \\ 0 & 1/6 & 0 \\ 0 & 0 & 1/6 \end{bmatrix}$$

(c) Since the gradient of a linear function is constant we can move the integrand outside the integral:

$$a_{ij} = \iint_K \nabla\lambda_j \cdot \nabla\lambda_i \, dx\, dy = (\nabla\lambda_j \cdot \nabla\lambda_i) \underbrace{\iint_K dx\, dy}_{\mu(K)} = \frac{1}{2}\left(\nabla\lambda_j \cdot \nabla\lambda_i\right).$$

From (a) we can compute: $\nabla\lambda_1 = (-1, -1)$, $\nabla\lambda_2 = (0, 1)$ and $\nabla\lambda_3 = (1, 0)$. We thus get: $\nabla\lambda_1 \cdot \nabla\lambda_1 = 2$, $\nabla\lambda_2 \cdot \nabla\lambda_2 = 1$, $\nabla\lambda_3 \cdot \nabla\lambda_3 = 1$, $\nabla\lambda_1 \cdot \nabla\lambda_2 = \nabla\lambda_2 \cdot \nabla\lambda_1 = -1$, $\nabla\lambda_1 \cdot \nabla\lambda_3 = \nabla\lambda_3 \cdot \nabla\lambda_1 = -1$ and $\nabla\lambda_2 \cdot \nabla\lambda_3 = \nabla\lambda_3 \cdot \nabla\lambda_2 = 0$.

The final result is therefore:

$$\begin{bmatrix} 1 & -1/2 & -1/2 \\ -1/2 & 1/2 & 0 \\ -1/2 & 0 & 1/2 \end{bmatrix}$$

$\square$