TMA225 Differential Equations and Scientific Computing, part A

Solutions to Problems Week 1

September 9, 2002

Week 1:

Problem 1. Use the expressions $\lambda_a(x) = \frac{b-x}{b-a}$ and $\lambda_b(x) = \frac{x-a}{b-a}$ to show that

$$\lambda_a(x) + \lambda_b(x) = 1; \quad a\,\lambda_a(x) + b\,\lambda_b(x) = x.$$

Give a geometrical interpretation by plotting $\lambda_a(x)$, $\lambda_b(x)$, $\lambda_a(x) + \lambda_b(x)$, $a\,\lambda_a(x)$, $b\,\lambda_b(x)$, $a\,\lambda_a(x) + b\,\lambda_b(x)$ in the same figure.
Solution: Direct calculation gives $\lambda_a(x) + \lambda_b(x) = \frac{b-x}{b-a} + \frac{x-a}{b-a} = 1$ and $a\,\lambda_a(x) + b\,\lambda_b(x) = a\,\frac{b-x}{b-a} + b\,\frac{x-a}{b-a} = x$. The functions for the case $a = 2$ and $b = 3$ are plotted in Figure 1. $\square$

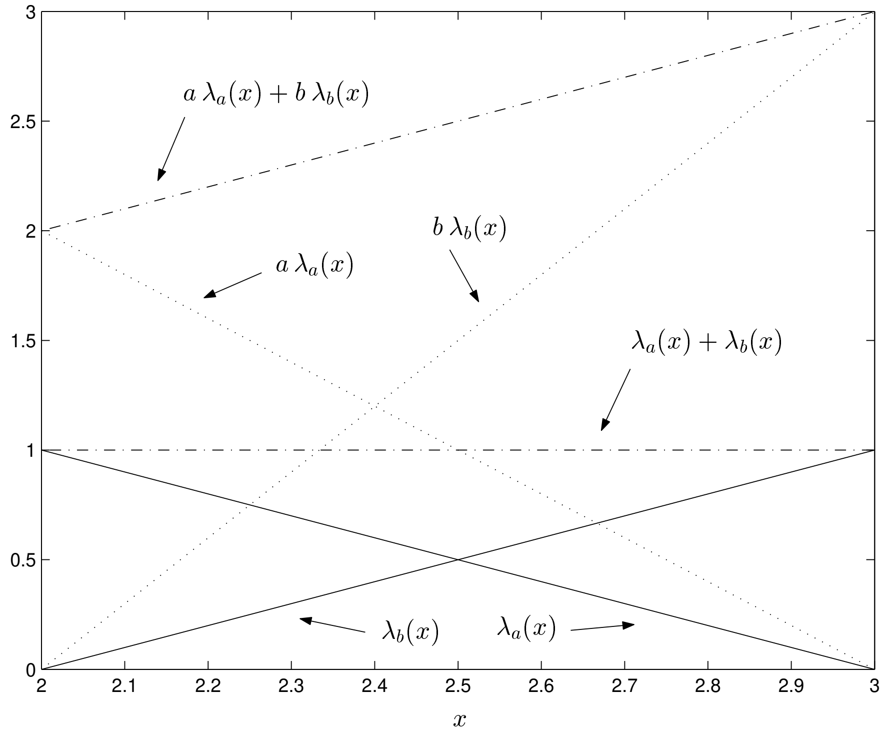

Figure 1: Problem 1 (Week 1). A plot of the functions.

Problem 2. Let $0 = x_0 < x_1 < x_2 < x_3 = 1$, where $x_1 = 1/6$ and $x_2 = 1/2$, be a partition of the interval $[0, 1]$ into three subintervals.
(a) Determine analytical expressions for the "hat-functions" φ_0, φ_1, φ_2, φ_3 in V_h (the space of continuous piecewise linear functions on this partition). Draw a figure.
(b) Which is the dimension of V_h?
(c) Plot the mesh function $h(x)$.
Solution:
(a) The "hat-functions" are given by the formula (with obvious modifications for φ_0 and φ_3):

$$\varphi_i(x) = \begin{cases} 0, & x \notin [x_{i-1}, x_{i+1}] \\ \frac{x - x_{i-1}}{x_i - x_{i-1}}, & x \in [x_{i-1}, x_i] \\ \frac{x_{i+1} - x}{x_{i+1} - x_i}, & x \in [x_i, x_{i+1}] \end{cases}$$

This gives

$$\varphi_0(x) = \begin{cases} 0, & x \notin [x_0, x_1] \\ 1 - 6x, & x \in [x_0, x_1] \end{cases} \quad , \quad \varphi_1(x) = \begin{cases} 0, & x \notin [x_0, x_2] \\ 6x, & x \in [x_0, x_1] \\ \frac{3-6x}{2}, & x \in [x_1, x_2] \end{cases}$$

and

$$\varphi_2(x) = \begin{cases} 0, & x \notin [x_1, x_3] \\ \frac{6x-1}{2}, & x \in [x_1, x_2] \\ 2 - 2x, & x \in [x_2, x_3] \end{cases} \quad , \quad \varphi_3(x) = \begin{cases} 0, & x \notin [x_2, x_3] \\ 2x - 1, & x \in [x_2, x_3] \end{cases},$$

where $x_0 = 0$, $x_1 = 1/6$, $x_2 = 1/2$ and $x_3 = 1$. See Figure 2.

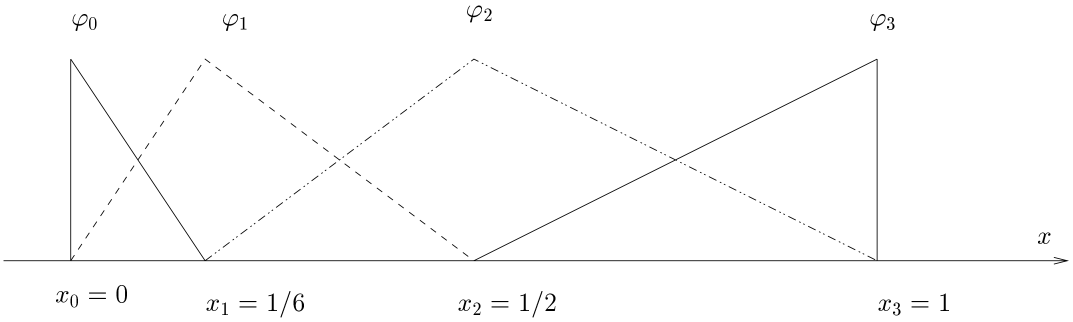

Figure 2: Problem 2(a) (Week 1). A plot of the "hat-functions".

(b) The dimension of V_h is equal to the number of basis functions which in this case is 4.
(c) See Figure 3. □

Problem 3. Let $f : [0, 1] \to \boldsymbol{R}$ be a Lipschitz continuous function. Determine the linear interpolant $\pi f \in \mathcal{P}(0, 1)$ and plot f and πf in the same figure, when
(a) $f(x) = x^2$,
(b) $f(x) = \sin(\pi x)$.
Solution: In general, the linear (nodal) interpolant $\pi f \in \mathcal{P}(x_0, x_1)$ can be written as

$$\pi f(x) = f(x_0)\,\varphi_0(x) + f(x_1)\,\varphi_1(x),$$

where $\varphi_i(x)$ form a basis of the space $\mathcal{P}(x_0, x_1)$ of linear polynomials on $I = [x_0, x_1]$. The x_i's are *nodes* where the interpolant's value is the same as the function's value. The basis functions we use are the "hat functions" $\varphi_i(x)$, $i = 0, 1$. Remember that $\varphi_0(x) = 1 - x$ and $\varphi_1(x) = x$, on $I = [0, 1]$.

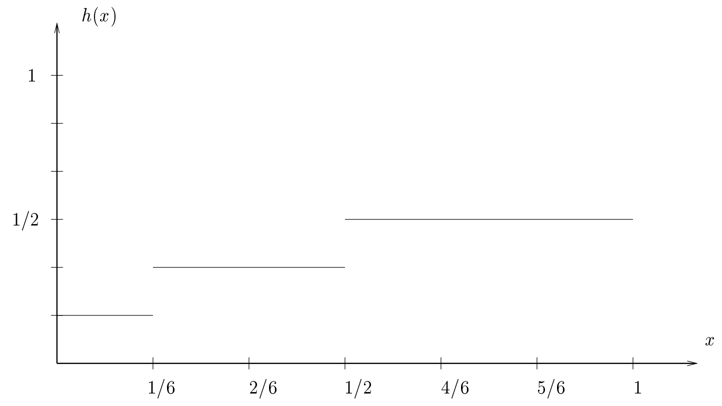

Figure 3: Problem 2(c) (Week 1). A plot of the mesh function.

(a) Therefore, the linear (nodal) interpolant of $f(x) = x^2$ on I can be written as

$$\begin{aligned}\pi f(x) &= \sum_{i=1}^{2} f(x_i)\varphi_i(x) \\ &= f(0)\,\varphi_0(x) + f(1)\,\varphi_1(x) \\ &= 0 \cdot (1-x) + 1 \cdot x \\ &= x\end{aligned}$$

(b) With $f(x) = \sin(\pi x)$, we analogously get

$$\begin{aligned}\pi f(x) &= \sum_{i=1}^{2} f(x_i)\varphi_i(x) \\ &= f(0)\,\varphi_0(x) + f(1)\,\varphi_1(x) \\ &= \sin(0)\,(1-x) + \sin(\pi)\,x \\ &= 0\end{aligned}$$

Obviously, the interpolant we've computed here is a poor approximation of $f(x) = \sin(\pi x)$. $\square$

Problem 4. Let $f : [0, 1] \to \boldsymbol{R}$ be a Lipschitz continuous function. Determine the continuous piecewise linear interpolant $\pi_h f \in V_h$, with $h(x)$ and V_h as in *Problem* 2 *(Week* 1*)*, and plot f and $\pi_h f$ in the same figure, when
(a) $f(x) = x^2$,
(b) $f(x) = \sin(\pi x)$.
Have we chosen a proper partition to approximate these functions? Can you think of a better one in case (a) and (b) if we are restricted to three subintervals?
Solution: The interval is partitioned according to $0 = x_0 < x_1 < x_2 < x_3 = 1$, with $x_1 = 1/6$ and $x_2 = 1/2$. On each subinterval we want the approximation, $\pi_h f(x)$, of $f(x)$

to be a straight line, $\alpha x + \beta$, and we want $\pi_h f(x)$ to *interpolate* $f(x)$ at the nodes $\{x_i\}_{i=0}^3$, i.e., $\pi_h f(x_i) = f(x_i)$, $i = 0, \ldots, 3$. (Note that this makes $\pi_h f(x)$ continuous also at the node-points.) This is accomplished by defining

$$\pi_h f(x) = f(x_0)\,\varphi_0(x) + f(x_1)\,\varphi_1(x) + f(x_2)\,\varphi_2(x) + f(x_3)\,\varphi_3(x)$$

where the "hat-functions" $\{\varphi_i\}_{i=0}^3$ have been computed in *Problem 2 (Week 1)*.
(a) With $f(x) = x^2$ we get

$$\begin{aligned}
\pi_h f(x) &= f(x_0)\,\varphi_0(x) + f(x_1)\,\varphi_1(x) + f(x_2)\,\varphi_2(x) + f(x_3)\,\varphi_3(x) \\
&= 0 \cdot \varphi_0(x) + \frac{1}{36} \cdot \varphi_1(x) + \frac{1}{4} \cdot \varphi_2(x) + 1 \cdot \varphi_3(x)
\end{aligned}$$

$$\left(\begin{aligned}
&= \begin{cases} 0 \cdot (1 - 6x) + \frac{1}{36} \cdot 6x, & x \in [0,\, 1/6] \\ \frac{1}{36} \cdot (\frac{3}{2} - 3x) + \frac{1}{4} \cdot (3x - \frac{1}{2}), & x \in [1/6,\, 1/2] \\ \frac{1}{4} \cdot (2 - 2x) + 1 \cdot (2x - 1), & x \in [1/2,\, 1] \end{cases} \\
&= \begin{cases} x/6, & x \in [0,\, 1/6] \\ -1/12 + 2x/3, & x \in [1/6,\, 1/2] \\ -1/2 + 3x/2, & x \in [1/2,\, 1] \end{cases}
\end{aligned} \right)$$

Remark. We are usually content with writing a function $v \in V_h$ in the form $v(x) = c_0\,\varphi_0(x) + c_1\,\varphi_1(x) + c_2\,\varphi_2(x) + c_3\,\varphi_3(x)$ which tells us that the nodal values are $(c_0,\, c_1,\, c_2,\, c_3)$ and that v is linear in between. Compare with plotting v in Matlab: `>> plot([x0 x1 x2 x3], [c0 c1 c2 c3])` connects the four points $(x_i,\, c_i)$ with straight lines. If we for some reason need to know the analytical expressions on each subinterval, they may of course be computed as above.

(b) For $f(x) = \sin(\pi x)$ we similarly get

$$\begin{aligned}
\pi_h f(x) &= f(x_0)\,\varphi_0(x) + f(x_1)\,\varphi_1(x) + f(x_2)\,\varphi_2(x) + f(x_3)\,\varphi_3(x) \\
&= 0 \cdot \varphi_0(x) + \frac{1}{2} \cdot \varphi_1(x) + 1 \cdot \varphi_2(x) + 0 \cdot \varphi_3(x)
\end{aligned}$$

$$\left(\begin{aligned}
&= \begin{cases} 0 \cdot (1 - 6x) + \frac{1}{2} \cdot 6x, & x \in [0,\, 1/6] \\ \frac{1}{2} \cdot (\frac{3}{2} - 3x) + 1 \cdot (3x - \frac{1}{2}), & x \in [1/6,\, 1/2] \\ 1 \cdot (2 - 2x) + 0 \cdot (2x - 1), & x \in [1/2,\, 1] \end{cases} \\
&= \begin{cases} 3x, & x \in [0,\, 1/6] \\ 3x/2 + 1/4, & x \in [1/6,\, 1/2] \\ 2 - 2x, & x \in [1/2,\, 1] \end{cases}
\end{aligned} \right)$$

$\square$

Problem 5. Let $h(x)$ be the mesh function for the partition defined in *Problem* 2 *(Week* 1*)*. Compute $||f - \pi_h f||_{L_\infty(0,\,1)}$ and $\frac{1}{8}||h^2 f''||_{L_\infty(0,\,1)}$, when

(a) $f(x) = x^2$,
(b) $f(x) = \sin(\pi x)$.
Hint: To compute $||f - \pi_h f||_{L_\infty(0,1)}$ you need to maximize the function $g(x) = |f(x) - \pi_h f(x)|$ on each subinterval and choose the largest of these three maxima. You can control your answers by also doing the computations with *Piecewise Polynomial Lab*.

If you think you have found better partitions in the end of *Problem* 4 *(Week* 1*)*, repeat the computations for these. Utilize *Piecewise Polynomial Lab*!
Solution: We have the mesh function

$$h(x) = \begin{cases} 1/6, & 0 < x < 1/6 \\ 1/3, & 1/6 < x < 1/2, \\ 1/2, & 1/2 < x < 1, \end{cases}$$

and want to compute $||f - \pi_h f||_{L_\infty(0,1)} = \max_{x\in[0,1]} |f(x) - \pi_h f(x)|$ and $\frac{1}{8}||h^2 f''||_{L_\infty(0,1)} = \frac{1}{8}\max_{x\in[0,1]} |h(x)^2 f''(x)|$.
(a) From *Problem* 4*(a) (Week* 1*)*, we have

$$|f(x) - \pi_h f(x)| = \begin{cases} |x^2 - x/6|, & 0 \le x \le 1/6, \\ |x^2 - (2x/3 - 1/12)\,|, & 1/6 \le x \le 1/2, \\ |x^2 - (3x/2 - 1/2)\,|, & 1/2 \le x \le 1. \end{cases}$$

Find maxima for each subinterval: (*Note:* Since $f''(x) = 2 > 0$, $f(x)$ is a *convex* function and the interpolant $\pi_h f(x)$ will therefore always be greater than $f(x)$. Further, since $f(x) - \pi_h f(x) = 0$ at the nodes, the local maxima will occur in the interior of the subintervals.)
$0 \le x \le 1/6:$

$$g(x) = |\underbrace{x^2 - \frac{x}{6}}_{\le 0,\ x\in[0,1/6]}| = \frac{x}{6} - x^2;$$

$$g'(x) = \frac{1}{6} - 2x = 0 \quad \Rightarrow \quad x = \frac{1}{12}; \qquad g''(x) = -2 \quad \Rightarrow \quad \textit{Maximum: } g(\frac{1}{12}) = \frac{1}{144}.$$

$1/6 \le x \le 1/2:$

$$g(x) = |\underbrace{x^2 - \left(\frac{2x}{3} - \frac{1}{12}\right)}_{\le 0,\ x\in[1/6,1/2]}| = \frac{2x}{3} - \frac{1}{12} - x^2;$$

$$g'(x) = \frac{2}{3} - 2x = 0 \quad \Rightarrow \quad x = \frac{1}{3}; \qquad g''(x) = -2 \quad \Rightarrow \quad \textit{Maximum: } g(\frac{1}{3}) = \frac{1}{36}.$$

$1/2 \le x \le 1:$

$$g(x) = |\underbrace{x^2 - \left(\frac{3x}{2} - \frac{1}{2}\right)}_{\le 0,\ x\in[1/2,1]}| = \frac{3x}{2} - \frac{1}{2} - x^2;$$

$$g'(x) = \frac{3}{2} - 2x = 0 \quad \Rightarrow \quad x = \frac{3}{4}; \qquad g''(x) = -2 \quad \Rightarrow \quad \textit{Maximum: } g(\frac{3}{4}) = \frac{1}{16}.$$

By comparing the three local maxima we get:

$$||f - \pi_h f||_{L_\infty(0,\,1)} = \frac{1}{16}.$$

Since

$$\frac{1}{8}|h(x)^2 f''(x)| = \begin{cases} \frac{2}{8\cdot 36} = \frac{1}{144}, & 0 < x < 1/6, \\ \frac{2}{8\cdot 9} = \frac{1}{36}, & 1/6 < x < 1/2, \\ \frac{2}{8\cdot 4} = \frac{1}{16}, & 1/2 < x < 1, \end{cases}$$

we also get

$$\frac{1}{8}||h^2 f''||_{L_\infty(0,\,1)} = \frac{1}{16}.$$

Remark. The reason we have equality in this case is that $f''(x) = 2$ is *constant.*

(b) From *Problem* 4*(b) (Week* 1*)*, we have

$$|f(x) - \pi_h f(x)| = \begin{cases} |\sin(\pi x) - 3x|, & 0 \le x \le 1/6, \\ |\sin(\pi x) - (3x/2 + 1/4)\,|, & 1/6 \le x \le 1/2, \\ |\sin(\pi x) - (2 - 2x)\,|, & 1/2 \le x \le 1. \end{cases}$$

Find maxima for each subinterval: (*Note:* Since $f''(x) = -\pi^2 \sin(\pi x) < 0$, for $x \in (0,\,1)$, $f(x)$ is *concave* on this interval and the interpolant $\pi_h f(x)$ will therefore be lesser than $f(x)$.)

$0 \le x \le 1/6:$

$$g(x) = |\underbrace{\sin(\pi x) - 3x}_{\ge 0,\ x \in [0,\,1/6]}| = \sin(\pi x) - 3x;$$

$$g'(x) = \pi\cos(\pi x) - 3 = 0 \quad \Rightarrow \quad x = \frac{1}{\pi}\arccos(\frac{3}{\pi}) \approx 0.096 \in [0,\ 1/6];$$

$$g''(x) = -\pi^2 \sin(\pi x) < 0 \quad \Rightarrow \quad \textit{Maximum:}\ g(\frac{1}{\pi}\arccos(\frac{3}{\pi})) \approx 0.009.$$

$1/6 \le x \le 1/2:$

$$g(x) = |\underbrace{\sin(\pi x) - \left(\frac{3x}{2} + \frac{1}{4}\right)}_{\ge 0,\ x \in [1/6,\,1/2]}| = \sin(\pi x) - \left(\frac{3x}{2} + \frac{1}{4}\right);$$

$$g'(x) = \pi\cos(\pi x) - \frac{3}{2} = 0 \quad \Rightarrow \quad x = \frac{1}{\pi}\arccos(\frac{3}{2\pi}) \approx 0.342 \in [1/6,\ 1/2];$$

$$g''(x) = -\pi^2 \sin(\pi x) < 0 \quad \Rightarrow \quad \textit{Maximum:}\ g(\frac{1}{\pi}\arccos(\frac{3}{2\pi})) \approx 0.116.$$

$1/2 \le x \le 1:$

$$g(x) = |\underbrace{\sin(\pi x) - (2 - 2x)}_{\ge 0,\ x \in [1/2,\,1]}| = \sin(\pi x) - (2 - 2x)\,;$$

$$g'(x) = \pi\cos(\pi x) + 2 = 0 \quad \Rightarrow \quad x = \frac{1}{\pi}\arccos(-\frac{2}{\pi}) \approx 0.720 \in [1/2,\, 1];$$

$$g''(x) = -\pi^2\sin(\pi x) < 0 \quad \Rightarrow \quad \textit{Maximum: } g(\frac{1}{\pi}\arccos(-\frac{2}{\pi})) \approx 0.211.$$

By comparing the three local maxima we get:

$$||f - \pi_h f||_{L_\infty(0,\,1)} = \sin(\arccos(-\frac{2}{\pi})) - 2(1 - \frac{1}{\pi}\arccos(-\frac{2}{\pi})) \approx 0.211.$$

Since

$$\frac{1}{8}|h(x)^2 f''(x)| = \begin{cases} \frac{\pi^2\sin(\pi x)}{8\cdot 36} \leq \frac{\pi^2\sin(\frac{\pi}{6})}{8\cdot 36} = \frac{\pi^2}{576}, & 0 < x < 1/6, \\ \frac{\pi^2\sin(\pi x)}{8\cdot 9} \leq \frac{\pi^2\sin(\frac{\pi}{2})}{8\cdot 9} = \frac{\pi^2}{72}, & 1/6 < x < 1/2, \\ \frac{\pi^2\sin(\pi x)}{8\cdot 4} \leq \frac{\pi^2\sin(\frac{\pi}{2})}{8\cdot 4} = \frac{\pi^2}{32}, & 1/2 < x < 1, \end{cases}$$

we get

$$\frac{1}{8}||h^2 f''||_{L_\infty(0,\,1)} = \frac{\pi^2}{32} \approx 0.308.$$

□