Assignment 2

Differential Equations and Scientific Computing Kb2, part A

General

This is the assignment for the second part (Week 5–8) of the course which covers the solution of two-dimensional boundary value problems using the finite element method.

You are asked to hand in a written documentation of your work. Remember that it is always important that your *written work is neat and easily accessible*.

Your work is individual and each student hands in their own documentation. However, it is natural that you work together in pairs of two students.

Problems marked by * are primarily intended for students aiming at higher grades (4 or 5). It is not compulsory to hand in any of the * problems, but you may of course do so if you like. Your grade will be based on your result on the written exam, and will not be affected by whether or not you hand in any * problems. Note, however, that some of the questions on the written exam will be based on * problems, so in order to achieve higher grades you need to work also with these problems.

Finally, please use the cover sheet included in the end of the assignment.

Deadline: Oct 22, 16.00 in the ASECO building

A. My solver for boundary value problems

1. Write a *finite element solver* for boundary value problems of the form

$$-\nabla \cdot a\nabla u + cu = f, \quad \text{in } \Omega,$$

with Robin boundary conditions

$$-n \cdot a\nabla u = \gamma(u - g_D) + g_N, \quad \text{on } \partial\Omega.$$

Start from the template program `MyFirst2DPoissonSolver`[1]. Make sure you understand the basic steps in the finite element main program as well as the details of the function `MyFirst2DPoissonAssembler`.

2. Choose a problem and illustrate the fact that when γ is large the Dirichlet condition $u = g_D$ on $\partial\Omega$ is imposed. Use for instance $\gamma = 1, 10, 100$, and 1000, and study the behavior of u at the boundary.

3. (a) Choose a problem with known exact solution and verify that your solver approximates the analytical solution when the mesh size decreases.

[1] Information about how to obtain this program will be given in the instructions for studio session 10: *MyFirst2DPoissonSolver*.

(b*) Write a program which calculates the energy norm or the L_2-norm of the error and verify the corresponding *a priori* error estimates using a log-log diagram.

4*. Extend your program to the *convection-diffusion* problem

$$-\nabla \cdot a\nabla u + b \cdot \nabla u + cu = f, \quad \text{in } \Omega.$$

Experiment with different diffusion coefficients $a = a(x) = a(x_1, x_2)$ and convective velocities $b = b(x) = b(x_1, x_2)$ to understand the behavior of the convection-diffusion problem.

5. Extend your solver to the *time dependent* problem[2]

$$\begin{aligned} u_t - \nabla \cdot a\nabla u + cu &= f, \quad \text{in } \Omega \times (0, T), \\ u(x, 0) &= u_0(x), \quad x \in \Omega, \\ -n \cdot a\nabla u &= \gamma(u - g_D) + g_N, \quad \text{on } \partial\Omega \times (0, T), \end{aligned}$$

where u_t denotes the time derivative.

6*. Include *adaptive mesh-refinement* based on an a posteriori error estimate for the stationary problem.

Hand in: a short documentation of your work.

B. Problems

1. Solve the problems handed out each week. Note that some of these problems also may be marked by *: cf. the remark in section *General*.

Hand in: – (do not hand in solutions to the problems)

C. Topics presentations

For each topic below, write a short summary, including basic ideas and theoretical results. Make sure you understand all basic steps. Working on these presentations gives you an overview of the theory, and is a good way to prepare yourself for the written exam.

Important: Note that both the one and two dimensional cases are now included. You may primarily consider the two dimensional case in your notes and indicate when there is a difference between one and two dimensions. Note also that some proofs (interpolation) are only required in the one dimensional case.

[2] We will do this in studio session 11: *Tidsberoende problem (2D)*.

1. (a) Triangulation of a two dimensional domain: mesh function, minimal angle, and data structures.
   (b)* Local mesh refinement algorithm.

2. The vector space of continuous piecewise linear polynomials V_h on a triangulation of a domain. Representation formula of functions $v \in V_h$ as a linear combination of tent basis functions.

3. Approximation of a given function by interpolation: definition and error estimates including proofs in the one dimensional case.

4. Approximation of given function by L_2-projection: definition, derivation of linear system of equations, algorithm, existence, uniqueness, and error estimate.

5. Approximation of integrals by numerical quadrature: basic examples and exactness for polynomials of certain orders and accuracy.

6. Approximation of solutions to differential equations by the finite element method: (a) formulation, derivation of discrete system of equations, algorithm, existence and uniqueness.
   (b) Basic steps in the finite element algorithm.
   (c*) Details of the assembly algorithm.

7. Boundary conditions: Neumann, Robin, and non-homogeneous Dirichlet conditions.

8. Formulate and prove an a priori error estimate in the energy norm for the finite element method with homogeneous Dirichlet boundary conditions.

9. (a) *Formulate* an a posteriori error estimate in the energy norm (with homogeneous Dirichlet boundary conditions). Compare with an a priori error estimate.
   (b*) *Prove* the a posteriori error estimate in the one dimensional case.

10. Formulate a time dependent method based on backward Euler.

11*. (a) Formulate and prove an a priori error estimate in the energy norm with Robin boundary conditions.
   (b) Formulate an a posteriori error estimate in the energy norm with Robin boundary conditions and prove your result in one spatial dimension.

12*. Explain what an adaptive algorithm is and how a posteriori error estimates are used in this context.

Hand in: a short readable summary of each topic (without too much detail).

D. Application

1. Use your finite element solver to solve the example application (handed out during LV6) or solve a problem of your own interest.

2*. Present a second application to a problem of interest.

Remember that you can solve stationary, time dependent, and also eigenvalue problems.

Hand in: a presentation of your work including the statement and motivation for the problem, mathematical modeling, and computational results.

Assignment 2

Name:

Date:

Coworker:

Problems solved

A1	A2	A3.a	A3.b*	A4*	A5	A6*

C1.a	C1.b*	C2	C3	C4	C5.a	C5.b	C5.c*	C6
C7	C8.a	C8.b*	C9	C10	C11.a*	C11.b*	C12*	–

D1	D2*