Applied Mathematics: Body & Soul
Vol I-III

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Preface

> I admit that each and every thing remains in its state until there is reason for change. (Leibniz)

The need of reform of mathematics education

Mathematics education needs to be reformed as we now pass into the new millenium. We share this conviction with a rapidly increasing number of researchers and teachers of mathematics and topics of science and engineering based on mathematical modeling. The reason is of course the computer revolution, which has fundamentally changed the possibilities of using mathematical and computational techniques for modeling, simulation and control of real phenomena. New products and systems may be developed and tested through computer simulation on time scales and at costs, which are orders of magnitude smaller than using traditional techniques based on extensive laboratory testing, hand calculations and trial and error.

At the heart of the new simulation techniques lie the new fields of Computational Mathematical Modeling CMM, including Computational Mechanics, Physics, Fluid Dynamics, Electromagnetics and Chemistry, all based on solving systems of differential equations using computers, combined with geometric modeling/Computer Aided Design CAD. Computational modeling is also finding revolutionary new applications in biology, medicine, environmental sciences, economy and finance markets.

The education in mathematics forms the basis of science and engineering education from undergraduate to graduate level, because engineering and science are largely based on mathematical modeling. The level and the quality of the mathematics education sets the level of the education as a whole. The new technology of CMM/CAD crosses borders between traditional engineering disciplines and schools, and puts strong forces to modernize engineering education in both content and form from basic to graduate level.

Our reform program

Our own reform work started some 20 years ago in courses in CMM on advanced undergraduate level, and has through the years successively penetrated through the system to the basic education in calculus and linear algebra. Our aim has become to develop a complete program for mathematics education in science and engineering from basic undergrauduate to graduate education. As of now our program contains the series of books:

1. *Computational Differential Equations*, (*CDE*)

2. *Applied Mathematics: Body* & *Soul I-III*, (*AM I-III*)

3. *Applied Mathematics: Body* & *Soul VI-*, (*AM IV-*).

AM I-III is the present book in three volumes I-III covering the basics of calculus and linear algebra. *AM IV-* offers a continuation with a series of volumes dedicated to specific areas of applications such as *Dynamical Systems (IV)*, *Fluid Mechanics (V)*, *Solid Mechanics (VI)* and *Electromagnetics (VII)*, which will start appearing in 2003. *CDE* published in 1996 may be be viewed as a first version of the whole *Applied Mathematics: Body* & *Soul* project.

Our program also contains a variety of software (collected in the *Mathematics Laboratory*), and by complementary material with step-by step instructions for self-study, problems with solutions, and projects, all freely available on-line. Our ambition is to be able to offer a "box" containing a set of books, software and additional instructional material, which can serve as a basis for a full applied mathematics program in science and engineering from basic to graduate level. Of course, we hope this to be an on-going project with new material being added gradually.

We have been running an applied mathematics program based on *AM* from first year for the students of chemical engineering at Chalmers since the Fall 99, and we have used collectedvarious advanced undergraduate courses

Main features of the program

- the program is based on a synthesis of mathematics, computation and application
- the program is based on new literature, giving a new unified presentation from start based on constructive mathematical methods including a computational methodology for differential equations
- the program contains, as an integrated part, software on different levels of complexity
- the student acquires solid skills of implementing computational methods and developing applications and software on a Matlab platform
- the synthesis of mathematics and computation opens the mathematics education to applications, and gives a basis for the effective use of modern mathematical methods in mechanics, physics, chemistry and applied subjects
- the synthesis building on constructive mathematics gives a synergetic effect allowing the study of complex systems already in the basic education, including the basic models of mechanical systems, heat conduction, wave propagation, elasticity, fluid flow, electro-magnetism, reaction-diffusion, molecular dynamcis and corresponding coupled multi-physics problems
- the program increases the motivation of the student by applying mathematical methods to interesting and important concrete problems already from the start
- emphasis may be put on problem solving, project work and presentation
- the program gives theoretical and computational tools and builds confidence
- the program contains most of the traditional material from basic courses in analysis and linear algebra
- the program includes much material often left out in traditional programs such as constructive proofs of all the basic theorems in analysis and linear algebra and advanced topics such as nonlinear systems of algebraic/differential equations
- emphasis is put on giving the student a solid understanding of basic mathematical concepts such as real numbers, Cauchy sequences, Lipschitz continuity, and constructive tools for solving algebraic/differential equations, together with an ability to utilize these tools in advanced applications such as molecular dynamics.

- the program may be run on different levels of ambition concerning both mathematical analysis and computation, while keeping a common basic core.

AM I-III in brief

Roughly speaking, *AM I-III* contains a synthesis of calculus and linear algebra including computational methods and a variety of applications. Emphasis is put on constructive/computational methods with the double aim of making the mathematics both understandable and useful. Our ambition is to introduce the student early to both (from the perspective of traditional education) advanced mathematical concepts (such as Lipschitz continuity, Cauchy sequence, contraction mapping, initial-value problem for systems of differential equations), and advanced applications such as Lagrangian mechanics, n-body systems, population models, elasticity and electrical circuits, with an approach based on constructive/computational methods.

The idea is thus that making the student comfortable with both advanced mathematical concepts and modern computational techniques, will open a wealth of possibilities of applying mathematics to problems of real interest. This is in contrast to the traditional education where the emphasis is usually put on a set of analytical techniques within a conceptual framework of more limited scope. For example: we lead already in the second quarter the student to write (in Matlab) his/her own solver for general systems of ordinary differential equations based on mathematically sound principles (high conceptual and computational level), while traditional education at the same time often focuses on training the student to master a bag of tricks for symbolic integration. We also teach the student some tricks to that purpose, but our over-all goal is different.

Constructive mathematics: Body & Soul

In our work we have been led to the conviction that the constructive aspects of calculus and linear algebra need to be strengthened. Of course, constructive and computational mathematics are closely related and the development of the computer has boosted computational mathematics in recent years. Mathematical modeling has two basic dual aspects: one symbolic and the other constructive-numerical, which reflect the duality between the infinite and the finite, or the continuum and the discrete. The two aspects have been closely intertwined throughout the development of modern science from the development of calculus in the work of Euler, Lagrange, Laplace and Gauss into the work of von Neumann in our time. For example, Laplace's monumental *Mécanique Céleste* in five volumes presents a symbolic calculus for a mathematical model of gravitation taking the form of Laplace's equation, together with massive numerical computations giving concrete information concerning the motion of the planets in our solar system.

However, beginning with the search for rigor in the foundations of calculus in the 19th century, a split between the symbolic and constructive aspects gradually developed. The split accelerated with the invention of the electronic computer in the 1940s, after which the constructive aspects were pursued in the new fields of numerical analysis and computing sciences, primarily developed outside departments of mathematics. The unfortunate result today is that symbolic mathematics and constructive-numerical mathematics by and large are separate disciplines and are rarely taught together. Typically, a student first meets calculus restricted to its symbolic form and then much later, in a different context, is confronted with the computational side. This state of affairs lacks a sound scientific motivation and causes severe difficulties in courses in physics, mechanics and applied sciences building on mathematical modeling.

Creating from start a synthesis of constructive and symbolic mathematics representing a synthesis of Body & Soul, opens new possibilities: with computational techniques available the students may become familiar with nonlinear systems of differential equations already in early calculus, with a wealth of applications. Another consequence is that the basics of calculus including concepts like real number, Cauchy sequence, convergence, fixed point iteration, contraction mapping, is lifted out of the wardrobe of mathematical obscurities into the real world with direct practical importance. In one shot one can make mathematics education both deeper, broader and lift it to a higher level. This idea is underlying the present book, which thus in the setting of a standard engineering program, contains all the basic theorems of calculus including the proofs normally taught only in special honours courses, together with advanced applications such as systems of nonlinear differential equations. We have found that this seemingly impossible program indeed works surprisingly well. Admittedly, this is hard to believe without making real life experiments. We hope the reader will feel encouraged to do so.

Lipschitz continuity and Cauchy sequences

The usual definition of the basic concepts of *continuity* and *derivative*, which is presented in most Calculus text books today, build on the concept of *limit*: a function of a real valued function $f(x)$ of a real variable x is said to be continuous at $\bar{x}$ if $\lim_{x\to\bar{x}} f(x) = f(\bar{x})$, and $f(x)$ is said to be differentiable at $\bar{x}$ with derivative $f'(\bar{x})$ if

$$\lim_{x\to\bar{x}} \frac{f(x) - f(\bar{x})}{x - \bar{x}}$$

exists and equals $f'(\bar{x})$. We use different definitions, where the concept of limit does not intervene: we say that $f(x)$ is Lipschitz continuous with Lipschitz constant L_f on the interval $[a, b]$ if for all $x, \bar{x} \in [a, b]$, we have

$$|f(x) - f(\bar{x})| \leq L_f|x - \bar{x}|.$$

Further, we say that $f(x)$ is differentiable at $\bar{x}$ with derivative $f'(\bar{x})$ if there is a constant $K_f(\bar{x})$ such that for all x close to $\bar{x}$

$$|f(x) - f(\bar{x}) - f'(\bar{x})(x - \bar{x})| \leq K_f(\bar{x})|x - \bar{x}|^2.$$

This means that we put somewhat more stringent requirements on the concepts of continuity and differentiability than is done in the usual definitions; more precisely, we impose *quantitative* measures in the form of the constants L_f and $K_f(\bar{x})$, whereas the usual definitions using limits are *purely qualitative*.

Using these more stringent definitions we avoid pathological situations, which can only be confusing to the student (in particular in the beginning), and as indicated, we avoid using the (difficult) concept of limit in a setting where in fact no limit processes are really taking place. Thus, we do not lead the student to definitions of continuity and differentiability suggesting that all the time the variable x is tending to some value $\bar{x}$, that is, that all the time some kind of (strange?) limit process is taking place. In fact, continuity expresses that the difference $f(x)-f(\bar{x})$ is small if $x-\bar{x}$ is small, and differentiability expresses that $f(x)$ locally is close to a linear function, and to express these facts we do not have to invoke any limit processes.

These are examples of our leading philosophy of giving Calculus a *quantitative form*, instead of the usual purely qualitative form, which we believe helps both understanding and precision. We believe the price to pay for these advantages is usually well worth paying, and the loss in generality are only some pathological cases of little interest. We can in a natural way relax our definitions for example to Hölder continuity, while still keeping the quantitative aspect, and thereby increase the pathology of the exceptional cases.

The usual definitions of continuity and differentiability strive for maximal generality, typically considered to be a virtue by a pure mathematician, which however has pathological side effects. With a constructive point of view the interesting world is the constructible world and maximality is not an important issue in itself.

Of course, we do not stay away from limit processes, but we concentrate on issues where the concept of limit really is central, most notably in defining the concept of a *real number* as the limit of a Cauchy sequence of rational numbers, and a solution of an algebraic or differential equation as the limit of a Cauchy sequence of approximate solutions. Thus, we give the concept of *Cauchy sequence* a central role, while maintaining a constructive approach always seeking a constructive process for generating a Cauchy sequence.

In standard Calculus texts, the concepts of Cauchy sequence and Lipschitz continuity are not used, believing them to be too difficult to be presented to freshmen, while the concept of real number is left undefined (seemingly believing that a freshman is so familiar with this concept from early life that no further discussion is needed). In contrast, in our constructive approach these concepts play a central role already from start, and in particular we spend a good deal of attention to the fundamental aspect of contructibilty of real numbers (viewed as possibly neverending decimal expansions).

We emphasize that taking a constructive approach does not make mathematical life more difficult in any important way, as is often claimed by the ruling mathematical school of formalists/logicists: All theorems of interest in Calculus and Linear Algebra survive, with possibly some small unessential modifications to keep the quantitative aspect and make the proofs more precise. As a result we are able to present basic theorems such as Contraction Mapping Principle, Implicit Function Theorem, Inverse Function Theorem, Convergence of Newton's Method, in a setting of several variables with complete proofs as a part of our basic Calculus, while these results in the standard curriculum are considered to be way to difficult for this level.

Proofs and Theorems

Most mathematics books including Calculus texts follow a theorem-proof style, where first a theorem is presented and then a corresponding proof is given. This is seldom appreciated very much by the students, who often have difficulties with the role and nature of the proof concept.

We usually turn this around and first present an line of thought leading to some result, and then we state a corresponding theorem as a summary of the hypothesis and the main result. We thus rather use a proof-theorem format. We believe this is in fact often more natural than the theorem-proof style, since by first presenting the line of thought the different ingredients like hypothesis may be introduced in a logical order. The proof will then be just like any other line of thought, where one successively derives consequences from some starting point using different hypothesis as one goes along. We hope this will help to eliminate the often perceived mystery of proofs, simply because the student will not be aware of the fact that that a proof is being presented; it will just a logical line of thought, like any logical line of thought in everyday life. Only when the line of thought is finished, one may go back and call it a proof, and in a theorem collect the main result arrived at including the required hypothesis. As a consequence, in the Latex version of the book we do use a theorem-environment, but not any proof-environment; the proof is just a logical line of thought preceeding a theorem collecting the hypothesis and the main result.

The Mathematics Laboratory

We have developed various pieces of software to support our program into what we refer to as the *Mathematics Laboratory.* Some of the software serves the purpose of illustrating mathematical concepts such as roots of equations, Lipschitz continuity, fixed point iteration, differentiability, the definition of the integral, basis calculus for functions of several varibles, other pieces are supposed to be used as models for the students own computer realizations, and finally some pieces are aimed at applications such as solvers for differential equations. New pieces are being added continuously. Our ambition is to also add different multi-media realizations of various parts of the material.

In our program the students get a training from start in using Matlab as a tool for computation. The development of the constructive mathematical aspects of the basic topics of real numbers, functions, equations, derivatives and integrals, goes hand in hand with experience of solving equations with fixed point iteration or Newton's method, quadrature, and numerical methods or differential equations. The students see from their own experience that abstract symbolic concepts have roots down into constructive computation, which also gives a direct coupling to applications and physical reality.

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My heart is sad and lonely
for you I sigh, dear, only
Why haven't you seen it
I'm all for you body and soul
(Green, Body and Soul)

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II Calculus in several dimensions

$\Delta u = \nabla \cdot \nabla u$

$\frac{\partial u}{\partial t} - \Delta u = f$

$\frac{\partial^2 u}{\partial t^2} - \Delta u = f$

$i\frac{\partial u}{\partial t} - \Delta u + Vu = 0$

$\frac{\partial u}{\partial t} + u \cdot \nabla u - \nu \Delta u = f$

$\nabla \times H = J, \quad \nabla \cdot B = 0, \quad B = \mu H.$ **778**

1
What is mathematics?

The question of the ultimate foundations and the ultimate meaning of mathematics remains open; we do not know in what direction it will find its final solution or whether a final objective answer may be expected at all. "Mathematizing" may well be a creative activity of man, like language or music, of primary originality, whose historical decisions defy complete objective rationalization. (Weyl)

1.1 Introduction

We start out by giving a very brief idea of the nature of mathematics and the role of mathematics in our society.

1.2 The modern world: automatized production and computation

The mass consumption of the *industrial society* is made possible by the *automatized mass production* of material goods such as food, clothes, housing, TV-sets, CD-players and cars. If these items had to be produced by hand, they would be the privileges of only a select few.

Analogously, the emerging *information society* is based on mass consumption of *automatized computation* by computers that is creating a new "virtual reality " and is revolutionizing technology, communication, admin-

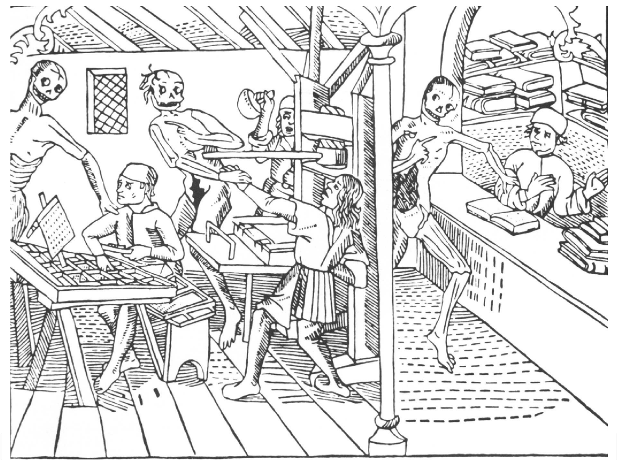

FIGURE 1.1. First picture of book printing technique (from Danse macabre, Lyon 1499)

istration, economy, medicine, and the entertainment industry. The information society offers immaterial goods in the form of knowledge, information, fiction, movies, music, games and means of communication. The modern PC or lap-top is a powerful computing device for mass production/consumption of information e.g. in the form of words, images, movies and music.

Key steps in the automatization or mechanization of production were: Gutenbergs's book printing technique (Germany, 1450), Christoffer Polhem's automatic machine for clock gears (Sweden, 1700), The Spinnning Jenny (England, 1764), Jacquard's punched card controlled weavingloom (France, 1801), Ford's production line (USA, 1913), see Fig. 1.1, Fig. 1.2, and Fig. 1.3.

Key steps in the automatization of computation were: Abacus (Ancient Greece, Roman Empire), Slide Rule (England, 1620), Pascals Mechanical Calculator (France, 1650), Babbage's Difference Machine (England, 1830), Scheutz' Difference Machine (Sweden, 1850), ENIAC Electronic Numerical Integrator and Computer (USA, 1945), and the Personal Computer PC (USA, 1980), see Fig. 1.5, Fig. 1.6, Fig. 1.7 and Fig. 1.8. The Difference Machines could solve simple differential equations and was used to compute tables of elementary functions such as the logarithm. ENIAC was one of the first modern computers (electronic and programmable), consisted of 18.000 vacuum tubes filling a room of 50×100 squarefeet with a weight of 30 tons and energy consuming of 200 kilowatts, and was used to solve the differential equations of ballistic firing tables as an important part of the Allied World War II effort.

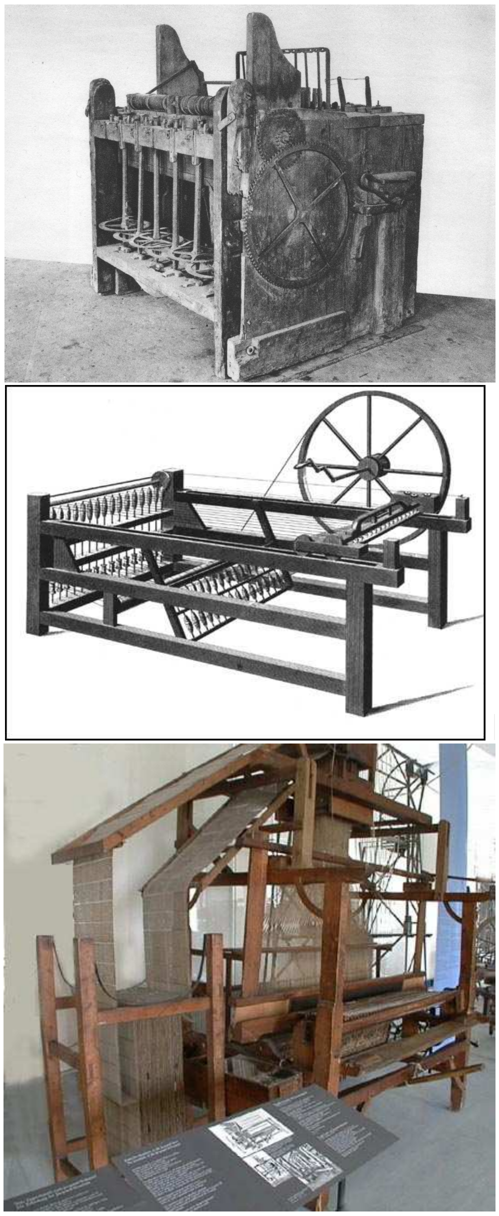

FIGURE 1.2. Christoffer Polhem's machine for clock gears (1700), Spinning Jenny (1764) and Jaquard's programmable loom (1801).

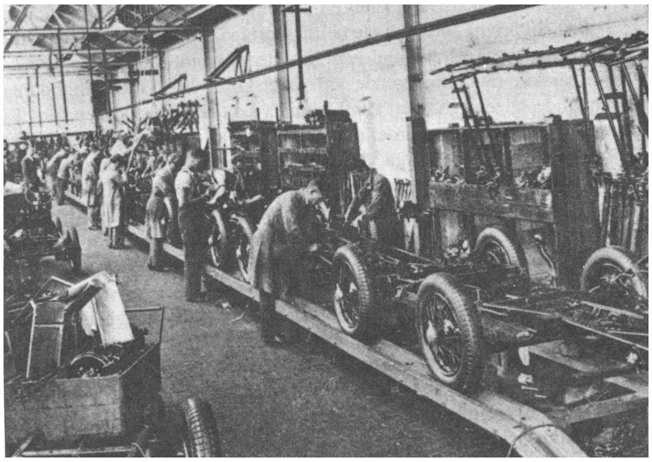

FIGURE 1.3. Ford assembly line (1913).

Automatization (or automation) is based on frequent repetion of a certain *algorithm* or scheme with new data at each repetition. The algorithm may consist of a sequence of relatively simple steps together creating a more complicated process. In automatized manufacturing, as in the production line of a car factory, physical material is modified following a strict repetitive scheme, and in automatized computation, the 1s and 0s of the microprocessor are modified billions of times each second following the computer program. Similarly, a *genetic code* of an organism may be seen as an algorithm that generates a living organism when realized in interplay with the environment. Realizing a genetic code many times (with small variations) generates populations of organisms. Mass-production is the key to increased complexity following the patterns of nature: elementary particle $\rightarrow$ atom $\rightarrow$ molecule and molecule $\rightarrow$ cell $\rightarrow$ organism $\rightarrow$ population, or the patterns of our society: individual $\rightarrow$ group $\rightarrow$ society or computer $\rightarrow$ computer network $\rightarrow$ global net.

1.3 The role of mathematics

Mathematics may be viewed as the language of computation and thus lies at the heart of the modern information society. Mathematics is also the language of science and thus lies at the heart of the industrial society that grew out of the *scientific revolution* in the 17th century that began when Leibniz and Newton created *Calculus*. Using Calculus, basic laws of mechanics and physics, such as Newton's law, could be formulated as *mathematical models* in the form of *differential equations*. Using the models, real phenomena

FIGURE 1.4. Computing device of the Inka Culture

could be *simulated* and controlled (more or less) and industrial processes could be created.

The mass consumption of both material and immaterial goods, considered to be a corner-stone of our modern democratic society, is made possible through automatization of production and computation. Therefore, mathematics forms a fundamental part of the technical basis of the modern society revolving around automatized production of material goods and automatized computation of information.

The vision of virtual reality based on automatized computation was formulated by Leibniz already in the 17th century and was developed further by Babbage with his Analytical Engine in the 1830s. This vision is finally being realized in the modern computer age in a synthesis of Body & Soul of Mathematics.

We now give some examples of the use of mathematics today that are connected to different forms of automatized computation.

1.4 Design and production of cars

In the car industry, a model of a component or complete car can be made using Computer Aided Design CAD. The CAD-model describes the ge-

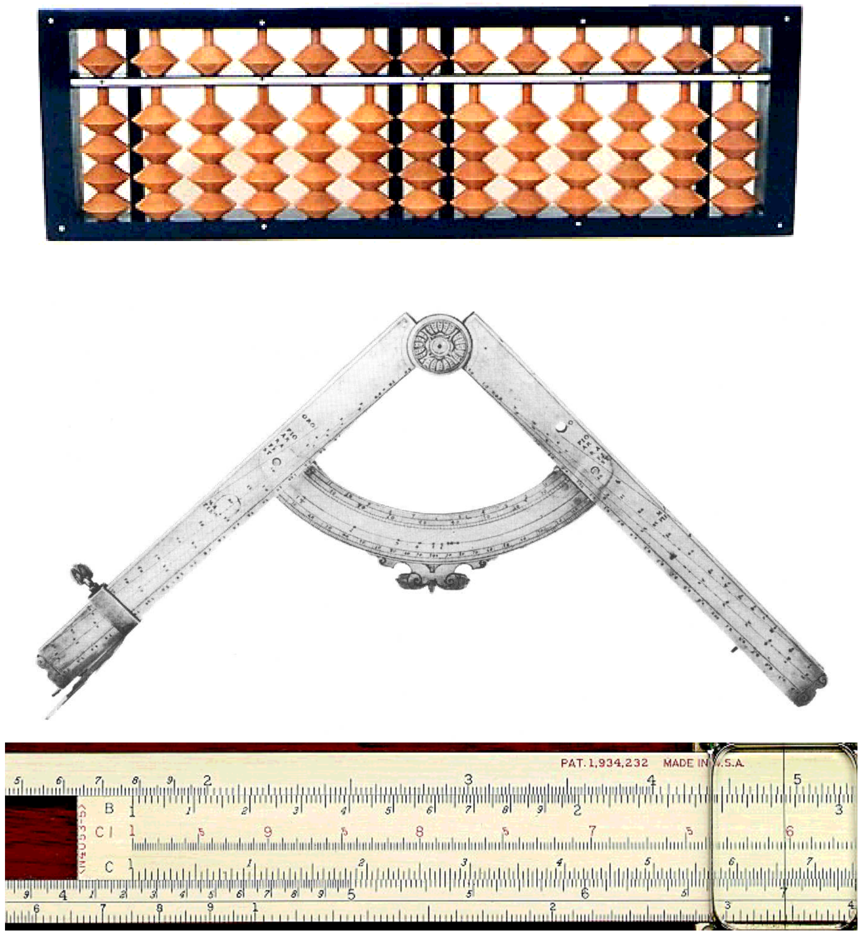

FIGURE 1.5. Classical computational tools: Abacus (300 B.C.-), Galileo's Compass (1597) and Slide Rule (1620-).

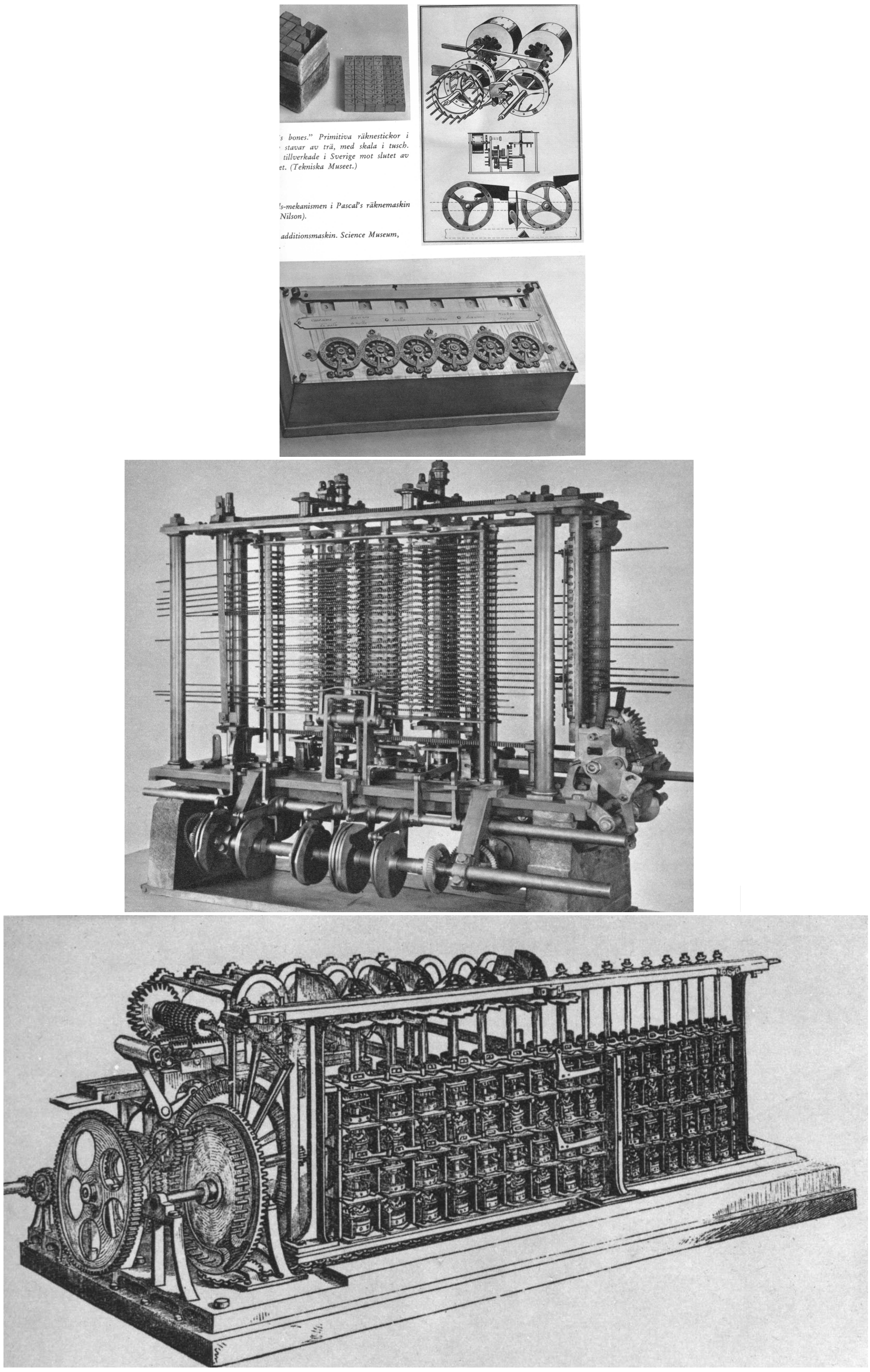

FIGURE 1.6. Napier's Bones (1617), Pascals Calculator (1630), Babbage's Difference Machine (1830) and Scheutz' Swedish Difference Machine (1850).

FIGURE 1.7. Odhner's mechanical calculator made in Göteborg, Sweden, 1919-1950

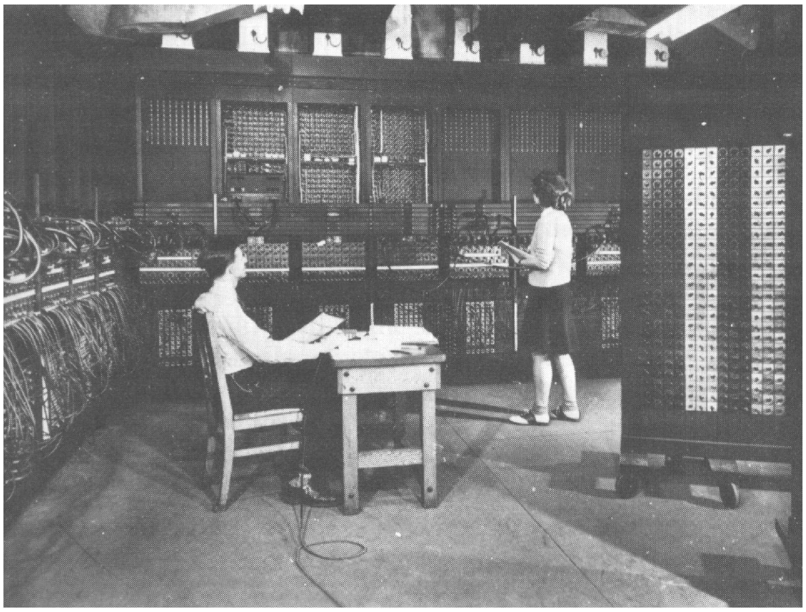

FIGURE 1.8. ENIAC Electronic Numerical Integrator and Calculator (1945)

ometry of the car through mathematical expressions and the model can be displayed on the computer screen. The performance of the component can then be tested in computer simulations, where differential equations are solved through massive computation, and the CAD-model is used as input of geometrical data. Further, the CAD data can be used in automatized production. The new technique is revolutionizing the whole industrial process from design to production.

1.5 Navigation: from stars to GPS

A primary force behind the development of geometry and mathematics since the Babylonians has been the need to navigate using information from the positions of the planets, stars, the Moon and the Sun. With a clock and a sextant and mathematical tables, the sea-farer of the 18th century could determine his position more or less accurately. But the results depended strongly on the precision of clocks and observations and it was easy for large errors to creep in. Historically, navigation has not been an easy job.

During the last decade, the classical methods of navigation have been replaced by GPS, the Global Positioning System. With a GPS navigator in hand, which we can buy for a couple of hundred dollars, we get our coordinates (latitude and longitude) with a precision of 50 meters at the press of a button. GPS is based on a simple mathematical principle known already to the Greeks: if we know our distance to three point is space with known coordinates then we can compute our position. The GPS uses this principle by measuring its distance to three satellites with known positions, and then computes its own coordinates. To use this technique, we need to deploy satellites, keep track of them in space and time, and measure relevant distances, which became possible only in the last decades. Of course, computers are used to keep track of the satellites, and the microprocessor of a hand-held GPS measures distances and computes the current coordinates.

The GPS has opened the door to mass consumption in navigation, which was before the privilege of only a few.

1.6 Medical tomography

The computer tomograph creates a pictures of the inside of a human body by solving a certain integral equation by massive computation, with data coming from measuring the attenuation of very weak X-rays sent through the body from different directions. This technique offers mass consumption of medical imaging, which is radically changing medical research and practice.

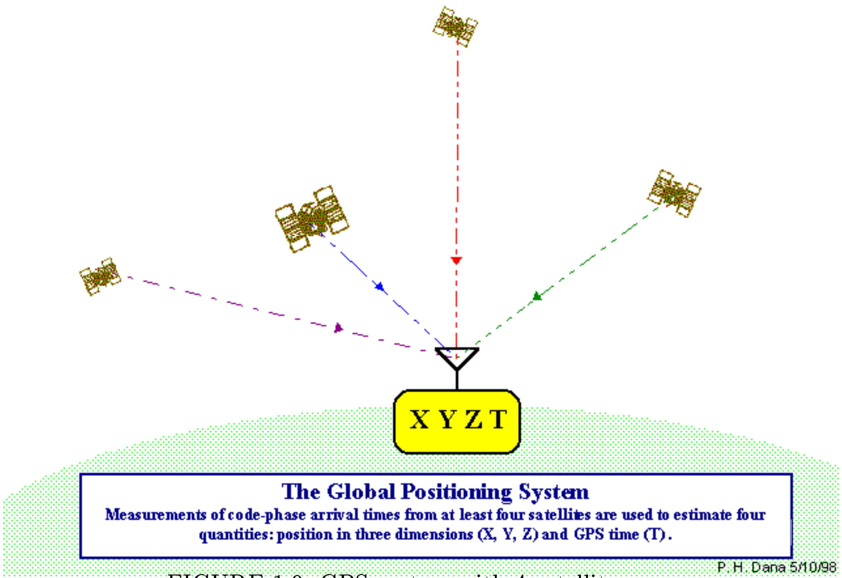

FIGURE 1.9. GPS-system with 4 satellites

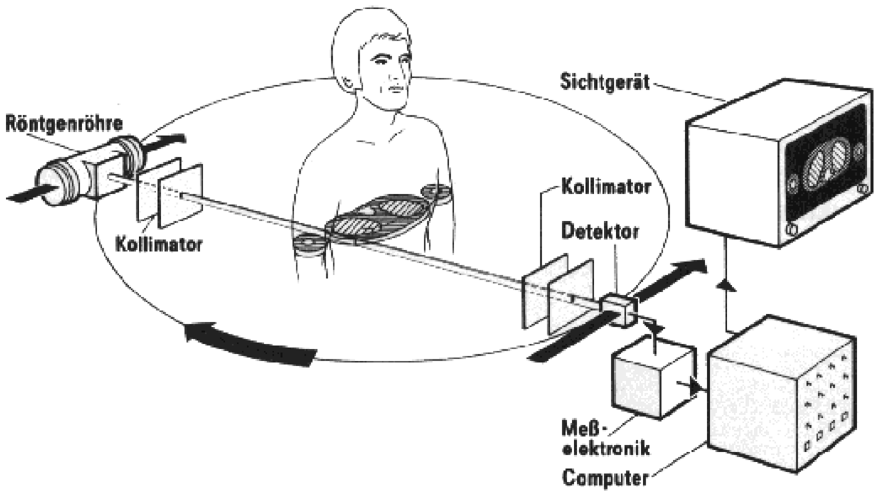

FIGURE 1.10. Medical tomograph.

1.7 Molecular dynamics and medical drug design

The classic way in which new drugs are discovered is an expensive and time-consuming process. First, a physical search is conducted for new organic chemical compounds, for example among the rain forests in South America. Once a new organic molecule is discovered, drug and chemical companies license the molecule for use in a broad laboratory investigation to see if the compound is useful. This search is conducted by expert organic chemists who build up a vast experience with how compounds can interact and which kind of interactions are likely to prove useful for the purpose of controlling a disease or fixing a physical condition. Such experience is needed to reduce the number of laboratory trials that are conducted, otherwise the vast range of possibilities is overwhelming.

The use of computers in the search for new drugs is rapidly increasing. One use is to makeup new compounds so as to reduce the need to make expensive searches in exotic locations like southern rain forests. As part of this search, the computer can also help classify possible configurations of molecules and provide likely ranges of interactions, thus greatly reducing the amount of laboratory testing time that is needed.

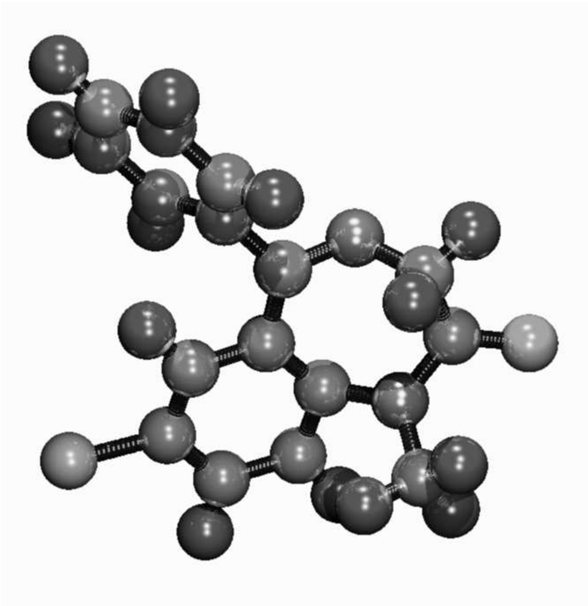

FIGURE 1.11. The Valium molecule

1.8 Weather prediction and global warming

Weather predictions are based on solving differential equations that describe the evolution of the atmosphere using a super computer. Reasonably reliable predictions of daily weather are routinely done for periods of a few days. For longer periods. the reliability of the simulation decreases rapidly, and with present day computers daily weather predictions for a period of two weeks are impossible.

However, forecasts over months of averages of temperature and rainfall are possible with present day computer power and are routinely performed.

Long-time simulations over periods of 20-50 years of yearly temperature-averages are done today to predict a possible *global warming* due to the use of fossil energy. The reliability of these simulations are debated.

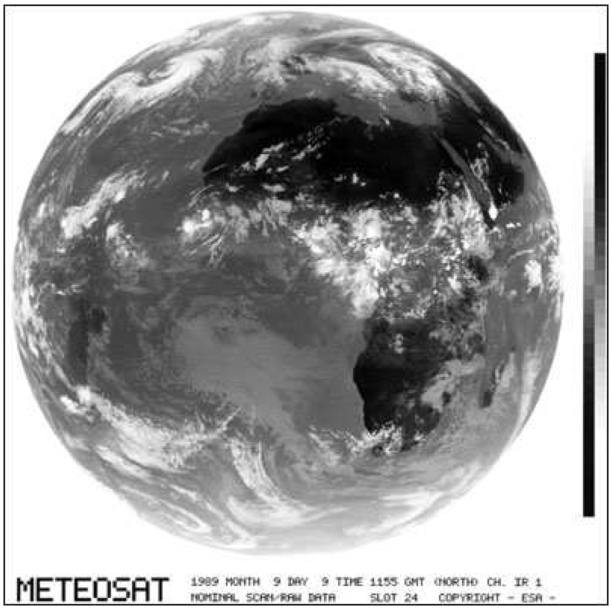

FIGURE 1.12. Clouds over West of Africa (ESA/Eumesat)

1.9 Economy: stocks and options

The Black-Scholes model for pricing options has created a new market of so called derivative trading as a complement to the stock market. To correctly price options is a mathematically complicated and computationally intensive task, and a stock broker with first class software for this purpose (which responds in a few seconds), has a clear trading advantage.

1.10 Languages

Mathematics is a *language*. There are many different languages. Our mother tongue, whatever it happens to be, English, Swedish, Greek, et cetera, is our most important language, which a child masters quite well at the age of three. To learn to write in our native language takes longer time and more effort and occupies a large part of the early school years. To learn to speak and write a foreign language is an important part of secondary education.

Language is used for *communication* with other people for purposes of cooperation, exchange of ideas or control. Communication is becoming increasingly important in our society as the modern means of communication develop.

Using a language we may create *models* of phenomena of interest, and by using models, phenomena may be studied for purposes of *understanding* or *prediction*. Models may be used for *analysis* focussed on a close examination of individual parts of the model and for *synthesis* aimed at understanding the interplay of the parts that is understanding the model as a whole. A *novel* is like a model of the real world expressed in a written language like English. In a novel the characters of people in the novel may be analyzed and the interaction between people may be displayed and studied.

The ants in a group of ants or bees in a bees hive also have a language for communication. In fact in modern biology, the interaction between cells or proteins in a cell is often described in terms of entities "talking to each other".

It appears that we as human beings use our language when we *think*. We then seem to use the language as a model in our head, where we try various possibilities in *simulations* of the real world: "If that happens, then I'll do this, and if instead that happens, then I will do so and so...". Planning our day and setting up our calender is also some type of modeling or simulation of events to come. Simulations by using our language thus seems to go on in our heads all the time.

There are also other languages like the language of musical notation with its notes, bars, scores, et cetera. A musical score is like a model of the real music. For a trained composer, the model of the written score can be very close to the real music. For amateurs, the musical score may say very little, because the score is like a foreign language which is not understood.

1.11 Mathematics as the language of science

Mathematics has been described as the language of science and technology including mechanics, astronomy, physics, chemistry, and topics like fluid and solid mechanics, electromagnetics et cetera. The language of mathematics is used to deal with *geometrical* concepts like *position* and *form* and *mechanical* concepts like *velocity*, *force* and *field*. More generally, mathematics serves as a language in any area that includes *quantitative* aspects described in terms of *numbers*, such as economy, accounting, statistics et cetera. Mathematics serves as the basis for the modern means of electronic *communication* where information is coded as sequences of 0's and 1's and is transferred, manipulated or stored.

The words of the language of mathematics often are taken from our usual language, like *points*, *lines*, *circles*, *velocity*, *functions*, *relations*, *transformations*, *sequences*, *equality*, *inequality* et cetera.

A mathematical word, term or concept is supposed to have a specific meaning defined using other words and concepts that are already defined. This is the same principle as is used in a Thesaurus, where relatively complicated words are described in terms of simpler words. To start the definition process, certain fundamental concepts or words are used, which cannot be defined in terms of already defined concepts. Basic relations between the fundamental concepts may be described in certain *axioms*. Fundamental concepts of Euclidean geometry are *point* and *line*, and a basic Euclidean axiom states that through each pair of distinct points there is a unique line passing. A *theorem* is a statement derived from the axioms or other theorems by using logical reasoning following certain rules of logic. The derivation is called a *proof* of the theorem.

1.12 The basic areas of mathematics

The basic areas of mathematics are

- Geometry
- Algebra
- Analysis.

Geometry concerns objects like *lines*, *triangles*, *circles*. Algebra and Analysis is based on *numbers* and *functions*. The basic areas of mathematics education in engineering or science education are

- Calculus
- Linear Algebra

Calculus is a branch of analysis and concerns properties of functions such as *continuity*, and operations on functions such as *differentiation* and *integration*. Calculus connects to Linear Algebra in the study of *linear functions* or linear transformations and to *analytical geometry*, which describes geometry in terms of numbers. The basic concepts of Calculus are

- function
- derivative
- integral.

Linear Algebra combines Geometry and Algebra and connects to Analytical Geometry. The basic concepts of Linear Algebra are

- vector
- vector space
- projection, orthogonality
- linear transformation

This book teaches the basics of Calculus and Linear Algebra, which are the areas of mathematics underlying most applications.

1.13 What is science?

The theoretical kernel of *natural science* may be viewed as having two components

- formulating equations (modeling),
- solving equations (computation).

Together, these form the essence of *mathematical modeling* and *computational mathematical modeling*. The first really great triumph of science and mathematical modeling is Newton's model of our planetary system as a set of differential equations expressing Newton's law connecting force, through the inverse square law, and acceleration. An *algorithm* may be seen as a strategy or constructive method to solve a given equation via computation. By applying the algorithm and computing, it is possible to simulate real phenomena and make predictions.

Traditional techniques of computing were based on symbolic or numerical computation with pen and paper, tables, slide ruler and mechanical calculator. Automatized computation with computers is now opening new possibilities of simulation of real phenomena according to Natures own principle of massive repetition of simple operations, and the areas of applications are quickly growing in science, technology, medicine and economics.

Mathematics is basic for both steps (i) formulating and (ii) solving equation. Mathematics is used as a language to formulate equations and as a set of tools to solve equations.

Fame in science can be reached by formulating or solving equations. The success is usually manifested by connecting the name of the inventor to the equation or solution method. Examples are legio: Newton's method, Euler's equations, Lagrange's equations, Poisson's equations, Laplace's equations, Navier's equations, Navier-Stokes' equations, Boussinesq's equations, Einstein's equations, Schrödinger's equations, Black-Scholes formula..., some of which we will meet below.

1.14 What is conscience?

The activity of the brain is believed to consist of electrical/chemical signals/waves connecting billions of synapses in some kind of large scale computation. The question of the nature of the *conscience* of human beings has played a central role in the development of human culture since the early Greek civilization, and today computer scientists seek to capture its evasive nature in various forms of Artificial Intelligence AI. The idea of a division of the activity of the brain into a (small) *conscious* "rational" part and a (large) *unconscious* "irrational" part, is widely accepted since the days of Freud. The rational part has the role of "analysis" and "control" towards some "purpose" and thus has features of Soul, while the bulk of the "computation" is Body in the sense that it is "just" electrical/chemical waves. We meet the same aspects in numerical optimization, with the optimization algorithm itself playing the role of Soul directing the computational effort towards the goal, and the underlying computation is Body.

We have been brought up with the idea that the conscious is in control of the mental "computation", but we know that this is often not the case. In fact, we seem to have developed strong skills in various kinds of after-rationalization: whatever happens, unless it is an "accident" or something "unexpected", we see it as resulting from a rational plan of ours made up in advance, thus turning a posteriori observations into a priori predictions.

1.15 How to come to grips with the difficulties of understanding the material of this book and eventually view it as a good friend

We conclude this introductory chapter with some suggestions intended to help the reader through the most demanding first reading of the book and reach a state of mind viewing the book as a good helpful friend, rather than the opposite. From our experience of teaching the material of this book, we know that it may evoke quite a bit of frustration and negative feelings, which is not very productive.

Mathematics is difficult: Choose your own level of ambition.

First, we have to admit that mathematics is a difficult subject, and we see no way around this fact.Secondly, one should realize that it is perfectly possible to live a happy life with a career in both academics and industry with only elementary knowledge of mathematics. There are many examples including Nobel Prize Winners. This means that it is adviceable to set a level of ambition in mathematics studies which is realistic and fits the interest profile of the individual student. Many students of engineering have other prime interests than mathematics, but there are also students who really like mathematics and theoretical engineering subjects using mathematics. The span of mathematical interest thus may be expected to be quite wide in a group of students following a course based on this book, and it seems reasonable that this would be reflected in the choice of level of ambition.

Advanced material: Keep an open mind and be confident.

The book contains quite a bit of material which is “advanced” and not usually met in undergraduate mathematics, and which one may bypass and still be completely happy. It is probably better to be really familiar with and understand a smaller set of mathematical tools and have the ability to meet new challenges with some self-confidence, than repeatedly failing to digest too large portions. Mathematics is so rich, that even a life of fully-time study can only cover a very small part. The most important ability must be to meet new material with an open mind and some confidence!

Some parts of matematics are easy.

On the other hand, there are many aspects of mathematics which are not so difficult, or even “simple”, once they have been properly understood. Thus, the book contains both difficult and simple material, and the first impression from the student may give overwhelming weight to the former. To help out we have collected the most essential nontrivial facts in short summaries in the form of *Calculus Tool Bag I and II*, *Linear Algebra Tool Bag*, *Differential Equations Tool Bag*, *Applications Tool Bag*, *Fourier Analysis Tool Bag* and *Analytic Functions Tool Bag*. The reader will find the tool bags surprisingly short: just a couple pages, altogether say 15-20 pages. If properly understood, this material carries a long way and is “all” one needs to remember from the math studies for further studies and professional activities in other areas. Since the book contains about 1200 pages it means

50-100 pages of book text for each one page of summary. This means that the book gives more than the absolute minimum of information and has the ambition to give the mathematical concepts a perspective concerning both history and applicability today. So we hope the student does not get turned off by the quite a massive number of words, by remembering that after all 15-20 pages captures the essential facts. During a period of study of say one year and a half of math studies, this effectively means about one third of a page each week!

Increased/decreased importance of mathematics

The book reflects both the increased importance of mathematics in the information society of today, and the decreased importance of much of the analytical mathematics filling the traditional curriculum. The student thus should be happy to know that many of the traditional formulas are no longer such a must, and that a proper understanding of relatively few basic mathematical facts can help a lot in coping with modern life and science.

Which chapters can I skip in a first reading?

We indicate by * certain chapters directed to applications, which one may by-pass in a first reading without loosing the main thread of the presentation, and return to at a later stage if desired.

Chapter 1 Problems

1.1. Find out which Nobel Prize Winners got the prize for formulating or solving equations.

1.2. Reflect about the nature of "thinking" and "computing".

1.3. Find out more about the topics mentioned in the text.

1.4. (a) Do you like mathematics or hate mathematics, or something in between? Explain your standpoint. (b) Specify what you would like to get out of your studies of mathematics.

1.5. Present some basic aspects of science.

FIGURE 1.13. Left person: "Isn't it remarkable that one can compute the distance to stars like Cassiopeja, Aldebaran and Sirius?". Right person: "I find it even more remarkable that one may know their names!" (Assar by Ulf Lundquist)

2
The Mathematics Laboratory

2.1 Introduction

This book is complemented by various pieces of software collected into the *Mathematics Laboratory*, freely available on the book web site. The *Mathematics Laboratory* contains different types of software organized under the following headings:

- *Math Experience*
- *Tools*
- *Applications*
- *Students Lab*

Math Experience serves the purpose of illustrating mathematical concepts from analysis and linear algebra such as roots of equations, Lipschitz continuity, fixed point iteration, differentiability, the definition of the integral, basis calculus for functions of several varibles.

Tools contains (i) *ready-mades* such as different solvers for differential equations and (ii) *shells* aimed at helping the student to make his own tools such as solvers of systems of equations and differential equations.

Applications contains more developed sowtware like *DOLFIN*, a platform for Dynamic Objectoriented Library for FINite Elements, *Tanganyika* for multiadaptive solution of systems of differential equations and *Mech-lab* and for simulating mechanical systems.

Many students may feel stimulated to develop their own applications during a mathematics program based on this book using the basis offered by the *Mathematics Laboratory*. We invite students to contribute applications to the *Students Lab*, which we hope may serve as a source of inspiration transferring know how from old to new students.

2.2 Math Experience

Math Experience is a collection of Matlab GUI software designed to offer a deeper understanding of important mathematical concepts and ideas such as, for example, convergence, continuity, linearization, differentiation, Taylor polynomials, integration, etc. The idea is to provide on-screen computer "labs" in which the student, by himself guided by a number of well designed questions, can seek to fully understand (a) the concepts and ideas as such and (b) the mathematical formulas and equations describing the concepts, by interacting with the lab environment in different ways. For example, in the Taylor lab (see Fig. 2.1) it is possible to give a function, or pick one from a gallery, and study its Taylor polynomial approximation of different degrees, how it depends on the point of focus by mouse-dragging the point, how it depends on the distance to the point by zooming in and out etc. There is also a movie where the terms in the Taylor polynomial are added one at a time. In the MultiD Calculus lab (see Fig. 2.2) it is possible to define a function $u(x_1, x_2)$ and compute its integral over a given curve or a given domain, to view its gradient field, contour plots, tangent planes etc. One may also study vector fields (u, v), view their divergence and rotation, compute the integrals of these quantities to verify the fundamental theorems of vector calculus, view the (u, v) mapped domain and the Jacobian of the map etc, etc.

Several labs of this type have been completed and will be made available from the book web page (see www.phi.chalmers.se), and more are under way according to the following list:

- Func lab - about relations and functions, inverse function etc.
- Graph Gallery - elementar functions and their parameter dependence.
- Cauchy lab - about sequences & convergence
- Lipschitz lab - the concept of continuity
- Root lab - about bisection and fixed point iteration
- Linearization and the derivative
- Newtons lab - illustrating Newtons method

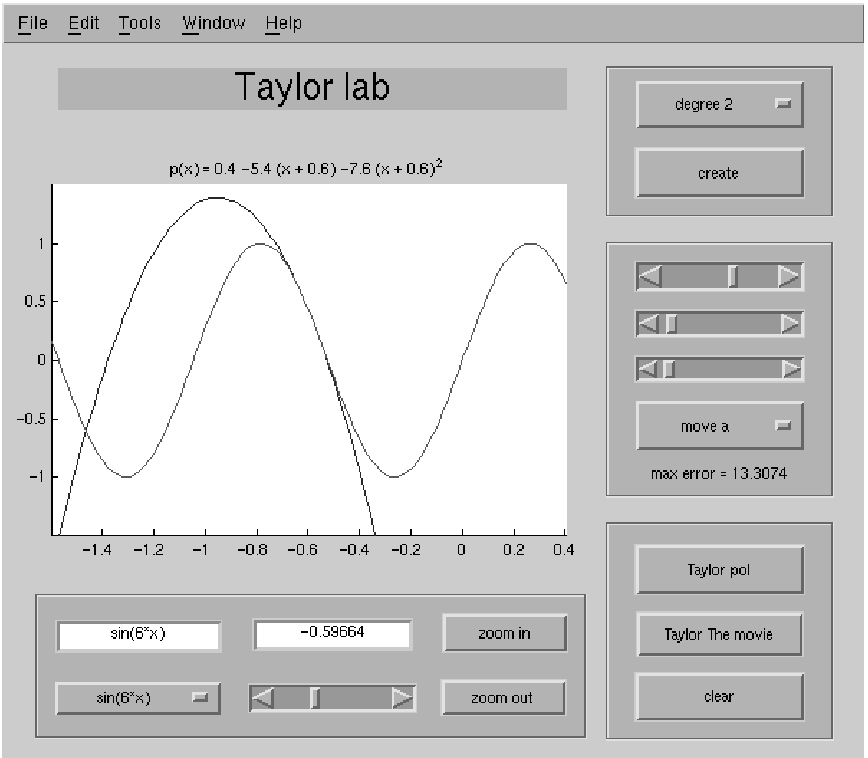

FIGURE 2.1. The Taylor lab

- Taylor lab - polynomials
- Opti lab - elementary optimization
- Piecewise polynomial lab - about piecewise polynomial approximation
- Integration lab - Euler and Riemann summation, adaptive integration
- Dynamical system lab
- Pendulum lab - the effect of linearization and approximation
- Vector algebra - a graphical vector calculator
- Analytic geometry - coupling geometry and linear algebra
- MultiD Calculus - integration and vector Calculus
- FE-lab - illustrating the finite element method
- Adaptive FEM - illustrating adaptive finite element techniques

- Poisson lab - fundamental solutions etc
- Fourier lab
- Wavelet lab
- Optimal control lab - control problems related to differential equations
- Archimedes lab - for experiments related to Archimedes principle

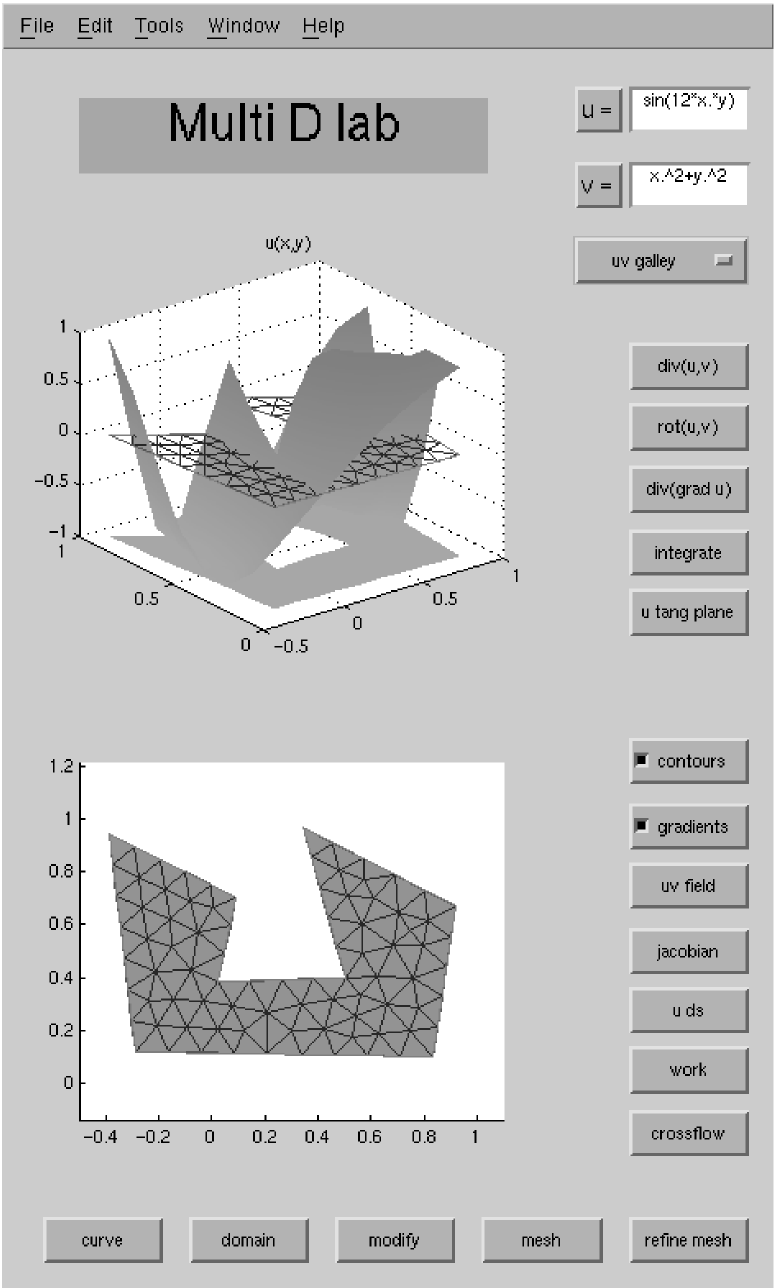

FIGURE 2.2. The MultiD Calculus lab

3
Introduction to modeling

> The best material model of a cat is another, or preferably the same, cat. (Rosenblueth/Wiener in Philosophy of Science 1945)

3.1 Introduction

We start by giving two basic examples of the use of mathematics for describing practical situations. The first example is a problem in household economy and the second is a problem in surveying, both of which have been important fields of application for mathematics since the time of the Babylonians. The models are very simple but illustrate fundamental ideas.

3.2 The Dinner Soup model

You want to make a soup for dinner together with your roommate, and following a recipe you ask your roommate to go to the grocery store and buy 10 dollars worth of potatoes, carrots, and beef according to the proportions 3:2:1 by weight. In other words, your roommate has 10 dollars to spend on the ingredients, which should be bought in the amounts so that by weight there are three times as much potatoes as beef and two times as much carrots as beef. At the grocery store, your roommate finds that potatoes are 1 dollar per pound, carrots are 2 dollars per pound, and beef is 8 dollars per pound. Your roommate thus faces the problem of figuring out how much of each ingredient to buy to use up the 10 dollars.

One way to solve the problem is by trial and error as follows: Your roommate could take quantities of the ingredients to the cash register in the proportions of 3:2:1 and let the clerk check the price, repeating until a total of 10 dollars is reached. Of course, both your roommate and the clerk could probably think of better ways to spend the afternoon. Another possibility would be to make a *mathematical model* of the situation and then seek to find the correct amounts to buy by doing some computations. The basic idea would be to use brains and pen and paper or a calculator, instead of labor intensive brute physical work.

The mathematical model may be set up as follows: Recalling that we want to determine the amounts of ingredients to buy, we notice that it is enough to determine the amount of beef, since we'll buy twice as much carrots as beef and three times as much potatoes as beef. Let's give a name to the quantity to determine. Let x denote the amount of meat in pounds to buy. The *symbol* x here represents an unknown quantity, or *unknown*, that we are seeking to determine by using available information.

If the amount of meat is x pounds, then the price of the meat to buy is $8x$ dollars by the simple computation

$$\text{cost of meat in dollars} = x\,\text{pounds} \times 8\,\frac{\text{dollars}}{\text{pound}} = 8x\,\text{dollars}.$$

Since there should be three times as much potatoes as meat by weight, the amount of potatoes in pounds is $3x$ and the cost of the potatoes is $3x$ dollars since the price of potatoes is one dollar per pound. Finally, the amount of carrots to buy is $2x$ and the cost is 2 times $2x = 4x$ dollars, since the price is 2 dollars per pound. The total cost of meat, potatoes and carrots is found by summing up the cost of each

$$8x + 3x + 4x = 15x.$$

Since we assume that we have 10 dollars to spend, we get the relation

$$15x = 10, \tag{3.1}$$

which expresses the equality of total cost and available money. This is an *equation* involving the unknown x and data determined by the physical situation. From this equation, your roommate can figure out how much beef to buy. This is done by dividing both sides of (3.1) by 15, which gives $x = 10/15 = 2/3 \approx 0.67$ pounds of meat. The amount of carrots should then be $2\times 2/3 = 4/3 \approx 1.33$, and finally the amount of potatoes $3\times 2/3 = 2$ pounds.

The *mathematical model* for this situation is $15x = 10$, where x is the amount of meat, $15x$ is the total cost and 10 is the available money. The modeling consists in expressing the total cost of the ingredients $15x$ in terms of the amount of beef x. Note that in this model, we only take into account what is essential for the current purpose of buying potatoes, carrots and meat for the Dinner Soup, and we did not bother to write down the prices of other items, like ice cream or beer. Determining the useful information is an important, and sometimes difficult, part of the mathematical modeling.

A nice feature of mathematical models is that they can be reused to simulate different situations. For example, if you have 15 dollars to spend, then the model $15x = 15$ arises with solution $x = 1$. If you have 25 dollars to spend, then the model is $15x = 25$ with solution $x = 25/15 = 5/3$. In general, if the amount of money y is given, then the model is $15x = y$. In this model we use the two symbols x and y, and assume that the amount of money y is given and the amount of beef x is an unknown quantity to be determined from the equation ($15x = y$) of the model. The roles could shift around: you may think of the amount of beef x as being given and the total cost or expenditure y to be determined (according to the formula $y = 15x$). In the first case, we would think of the amount of beef x as a function of the expenditure y and in the second the expenditure y as a function of x.

Assigning symbols to relevant quantities, known or unknown, is an important step in setting up a mathematical model of something. The idea of assigning symbols for unknown quantities was used already by the Babylonians (who had frequent use of models like the Dinner Soup model in organizing the feeding of the many people working on their irrigation systems).

Suppose that we could not solve the equation $15x = 10$, because of a lack of skill in solving equations (we may have forgotten the trick of dividing by 15 that we learned in school). We could then try to get a solution by some kind of trial and error strategy as follows. First we assume that $x = 1$. We then find that the total cost is 15 dollars, which is too much. We then try with a smaller quantity of meat, say $x = 0.6$, and compute the total cost to 9 dollars, which is too little. We then try with something between 0.6 and 1, say $x = 0.7$ and find that the cost would be 10.5 dollars, which is a little too much. We conclude that the right amount must be somewhere between 0.6 and 0.7, probably closer 0.7. We can continue in the same way to find as many decimals of x as we like. For instance we check next in the same way that x must be some where between 0.66 and 0.67. In this case we know the exact answer $x = \frac{2}{3} = 0.66666....$ The trial and error strategy just described is a model of the process of bringing food to the counter and letting the cashier compute the total prize. In the model we compute the prize ourselves without having to physically collect the items and bring them to the counter, which simplifies the trial and error process.

3.3 The Muddy Yard model

One of the authors owns a house with a 100m × 100m backyard that has the unfortunate tendency to form a muddy lake every time it rains. We show a perspective of the field on the left in Fig. 3.1. Because of the grading in

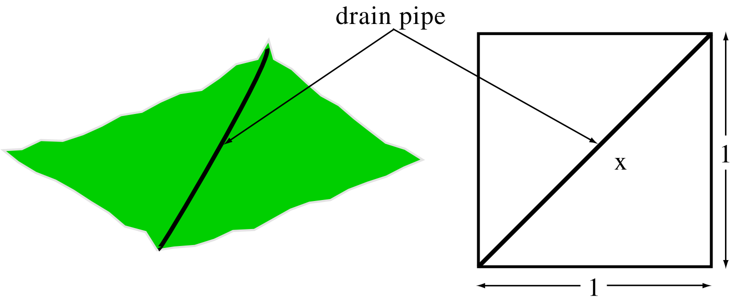

FIGURE 3.1. Perspective of a field with poor drainage and a model describing the dimensions.

the yard, the owner has had the idea for some time to fix this by digging a shallow ditch down the diagonal of the yard, laying some perforated plastic drain pipe, then covering the pipe back up. He is then faced with the problem of determining the amount of pipe that he needs to purchase. Since a survey of the property only provides the outside dimensions and the locations of corners and physically measuring the diagonal through the mud is not easy to do, he has decided to try to compute the distance using mathematics. Can mathematics help him in this endeavor?

Inspection of the property and a map indicates that the yard can be modeled as a horizontal square (the grading seems small), and we thus seek to compute the length of the diagonal of the square. We display the model on the right in Fig. 3.1, where we change to units of 100m, so the field is 1×1, and denote the length of the diagonal by x. We now recall Pythagoras' theorem, which states that $x^2 = 1^2 + 1^2 = 2$. To find the length x of the drain pipe, we are thus led to solve the equation

$$x^2 = 2. \tag{3.2}$$

Solving the equation $x^2 = 2$ may seem to be deceptively simple at first; the positive solution is just $x = \sqrt{2}$ after all. But walking into a store and asking for $\sqrt{2}$ units of pipe may not get a positive response. Precut pipes do not come in lengths calibrated by $\sqrt{2}$, neither do measure sticks indicate $\sqrt{2}$, and a clerk is thus going to need some concrete information about the value of $\sqrt{2}$ to be able to measure out a proper piece of pipe.

We can try to pin down the value of $\sqrt{2}$ by using a trial and error strategy. We can check easily that $1^2 = 1 < 2$ while $2^2 = 4 > 2$. So we know that

$\sqrt{2}$, whatever it is, is between 1 and 2. Next we can check $1.1^2 = 1.21$, $1.2^2 = 1.44$, $1.3^2 = 1.69$, $1.4^2 = 1.96$, $1.5^2 = 2.25$, $1.6^2 = 2.56$, $1.7^2 = 2.89$, $1.8^2 = 3.24$, $1.9^2 = 3.61$. Apparently $\sqrt{2}$ is between 1.4 and 1.5. Next we can try to fix the third decimal. Now we find that $1.41^2 = 1.9881$ while $1.42^2 = 2.0164$. So apparently $\sqrt{2}$ is between 1.41 and 1.42 and likely closer to 1.41. It appears that proceeding in this way, we can determine as many decimal places of $\sqrt{2}$ as we like, and we may consider the problem of computing how much drain pipe to buy to be solved!

Below we will meet many equations that have to be solved by using some variation of a trial and error strategy. In fact, most mathematical equations cannot be solved exactly by some algebraic manipulations, as we could do (if we were sufficiently clever) in the case of the Dinner Soup model (3.1). Consequently, the trial and error approach to solving mathematical equations is fundamentally important in mathematics. We shall also see that trying to solve equations such as $x^2 = 2$ carries us directly into the very heart of mathematics, from Pythagoras and Euclid through the quarrels on the foundations of mathematics that peaked in the 1930s and on into the present day of the modern computer.

3.4 A system of equations: Dinner Soup/Ice Cream model

Suppose you would like to finish off the Dinner Soup with some ice cream dessert at the cost of 3 dollars a pound, still at the total expense of 10 dollars. How much of each item should now be bought?

Well, if the amount ice cream is y pounds, the total cost will be $15x + 3y$ and thus we have the equation $15x + 3y = 10$ expressing that the total cost is equal to the available money. We now have two unknowns x and y, and we need one more equation. So far, we would be able to set $x = 0$ and solve for $y = \frac{10}{3}$ spending all the money on ice cream. This would go against some principle we learned as small kids. The second equation needed could come from some idea of balancing the amount of ice cream (junk food) to the amount of carrots (healthy food), for example according to the formula $2x = y + 1$, or $2x - y = 1$. Altogether, we would thus get the following system of two equations in the two unknowns x and y:

$$\begin{aligned} 15x + 3y &= 10, \\ 2x - y &= 1. \end{aligned}$$

Solving for y in the second equation, we get $y = 2x - 1$, which inserted into the first equation gives

$$15x + 6x - 3 = 10, \quad \text{that is} \quad 21x = 13, \quad \text{that is,} \quad x = \frac{13}{21}.$$

Finally, inserting the value of $x = \frac{13}{21} \approx 0.60$ into the equation $2x - y = 1$, we get $y = \frac{5}{21} \approx 0.24$, and we have found the solution of the system of equations modeling the present situation.

3.5 Formulating and solving equations

Let us put the Dinner Soup and Muddy Yard models into the perspective of formulating and solving equations presented above. Formulating equations corresponds to collecting information in systematic form, and solving equations corresponds to drawing conclusions from the collected information.

We began by describing the physical situations in the Dinner Soup and Muddy Yard models in terms of mathematical equations. This aspect is not just mathematical but involves also whatever knowledge from physics, economy, history, psychology, etcetera that may be relevant to describe the situation to be modeled. The equations we obtained in the Dinner Soup and the Muddy Yard models, namely $15x = 10$ and $x^2 = 2$, are examples of *algebraic equations* in which the data and the unknown x are both numbers. As we consider more complicated situations, we will often encounter models in which the data and the unknown quantities are *functions*. Such models typically contain derivatives and integrals and are then referred to as *differential equations* or *integral equations*.

The second aspect is *solving* the equations of the model to determine the unknown and gain new information about the situation at hand. In the case of the Dinner Soup model, we can solve the model equation $15x = 10$ exactly and express the solution $x = 2/3$ as a rational number. In the Muddy Yard model we resort to an iterative "trial and error" strategy to compute as many digits of the decimal expansion of the solution $x = \sqrt{2}$ as we may need. So it goes in general: once in a while, we can write down a solution of a model equation explicitly, but most often we have to be content with an approximate solution, the digits of which we can determine through some iterative computational process.

Note that there is no reason to be disappointed over the fact that equations representing mathematical models cannot be solved exactly, since the mathematical model is an approximation anyway, in general. It is better to have a complicated but accurate mathematical model equation, that admits only approximate solutions, than to have a trivial inaccurate model equation that can be solved exactly! The wonderful thing with computers is that they may compute accurate solutions also to complicated accurate model equations, which makes it possible to simulate real phenomena.

Chapter 3 Problems

3.1. Suppose that the grocery store sells potatoes for 40 cents per pound, carrots for 80 cents per pound, and beef for 40 cents per *ounce*. Determine the model relation for the total price.

3.2. Suppose that you change the soup recipe to have equal amounts of carrots and potatoes while the weight of these combined should be six times the weight of beef. Determine the model relation for the total price.

3.3. Suppose you go all out and add onions to the soup recipe in the proportion of $2:1$ to the amount of beef, while keeping the proportions of the other ingredients the same. The price of onions in the store is \$1 per pound. Determine the model relation for the total price.

3.4. While flying directly over the airport in a holding pattern at an altitude of 1 mile, you see your high rise condominium from the window. Knowing that the airport is 4 miles from your condominium and pretending that the condominium has height 0, how far are you from home and a cold beer?

3.5. Devise a model of the draining of a yard that has three sides of approximately the same length 2 assuming that we drain the yard by laying a pipe from one corner to the midpoint of the opposite side. What quantity of pipe do we need?

3.6. A father and his child are playing with a teeter-totter which has a seatboard 12 feet long. If the father weighs 170 pounds and the child weighs 45 pounds, construct a model for the location of the pivot point on the board in order for the teeter-totter to be in perfect balance? Hint: recall the principle of the lever which says that the products of the distances from the fulcrum to the masses on each end of a lever must be equal for the lever to be in equilibrium.

4
A very short Calculus course

Mathematics has the completely false reputation of yielding infallible conclusions. Its infallibility is nothing but identity. Two times two is not four, but it is just two times two, and that is what we call four for short. But four is nothing new at all. And thus it goes on in its conclusions, except that in the height the identity fades out of sight. (Goethe)

4.1 Introduction

Following up on the general idea of science as a combination of formulating and solving equations, we describe the bare elements of this picture from a mathematical point of view. We want to give a brief glimpse of the main themes of Calculus that will be discovered as we work through these books. In particular, we will encounter the magical words of *function*, *derivative*, and *integral*. If you have some idea of these concepts already, you will understand some of the outline. If you have no prior acquaintance with these concepts, you can use this section to just get a first taste of what Calculus is all about without expecting to understand the details at this point. Keep in mind that this is just a glimpse of the actors behind the curtain before the play begins!

We hope the reader can use this chapter to get a grip on the essence of Calculus by reading just a couple of pages. But this is really impossible in some sense because calculus contains so many formulas and details that it is easy to get overwhelmed and discouraged. Thus, we urge the reader to browse through the following couple of pages to get a quick idea and then return later and confirm with an "of course".

On the other hand, the reader may be surprised that something that is seemingly explained so easily in a couple of pages, actually takes several hundred pages to unwind in this book (and other books). We don't seem to be able give a good explanation of this "contradiction" indicating that "what looks difficult may be easy" and vice versa. We also present short summaries of Calculus in Chapter *Calculus Tool Bag I Calculus Tool Bag II*, which support the idea that a destillated essence of Calculus indeed can be given in a couple of pages.

4.2 Algebraic equations

We will consider *algebraic equations* of the form: find $\bar{x}$ such that

$$f(\bar{x}) = 0, \tag{4.1}$$

where $f(x)$ is a *function* of x. Recall that $f(x)$ is said to be a function of x if for each number x there is a number $y = f(x)$ assigned. Often, $f(x)$ is given by some algebraic formula: for example $f(x) = 15x - 10$ as in the Dinner Soup model, or $f(x) = x^2 - 2$ as in the Muddy Yard model.

We call $\bar{x}$ a *root* of the equation $f(x) = 0$ if $f(\bar{x}) = 0$. The root of the equation $15x - 10 = 0$ is $\bar{x} = \frac{2}{3}$. The positive root $\bar{x}$ of the equation $x^2 - 2 = 0$ is equal to $\sqrt{2} \approx 1.41$. We will consider different methods to compute a root $\bar{x}$ satisfying $f(\bar{x}) = 0$, including the trial and error method briefly presented above in the context of the Muddy Yard Model.

We will also meet *systems of algebraic equations*, where we seek to determine several unknowns satisfying several equations, as for the Dinner Soup/Ice cream model above.

4.3 Differential equations

We will also consider the following *differential equation*: find a function $x(t)$ such that for all t

$$x'(t) = f(t), \tag{4.2}$$

where $f(t)$ is a given function, and $x'(t)$ is the *derivative* of the function $x(t)$. This equation has several new ingredients. First, we seek here a *function* $x(t)$ with a set of different values $x(t)$ for different values of the variable t, and not just one single value of x like the root the algebraic equation $x^2 = 2$ considered above. In fact, we met this already in the Dinner Soup problem in case of a variable amount of money y to spend, leading to the equation $15x = y$ with solution $x = \frac{y}{15}$ depending on the variable y, that is,

$x = x(y) = \frac{y}{15}$. Secondly, the equation $x'(t) = f(t)$ involves the derivative $x'(t)$ of $x(t)$, so we have to investigate derivatives.

A basic part of Calculus is to (i) explain what a derivative is, and (ii) solve the differential equation $x'(t) = f(t)$, where $f(t)$ is a given function. The solution $x(t)$ of the differential equation $x'(t) = f(t)$, is referred to as an *integral* of $f(t)$, or alternatively as a *primitive function* of $f(t)$. Thus, a basic problem of Calculus is to find a primitive function $x(t)$ of a given function $f(t)$ corresponding to solving the differential equation $x'(t) = f(t)$.

We now attempt to explain (i) the meaning of (4.2) including the meaning of the derivative $x'(t)$ of the function $x(t)$, and (ii) give a hint at how to find the solution $x(t)$ of the differential equation $x'(t) = f(t)$ in terms of the given function $f(t)$.

As a concrete illustration, let us imagine a car moving on a highway. Let t represent *time*, let $x(t)$ be the *distance* traveled by the car at time t, and let $f(t)$ be the *momentary velocity* of the car at time t, see Fig. 4.1.

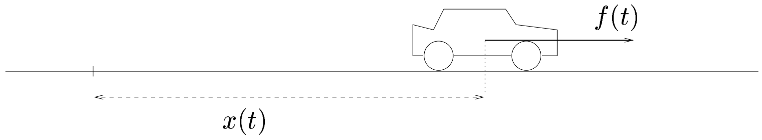

FIGURE 4.1. Highway with car (Volvo?) with velocity $f(t)$ and travelled distance $x(t)$.

We choose a starting time, say $t = 0$ and a final time, say $t = 1$, and we watch the car as it passes from its initial position with $x(0) = 0$ at time $t = 0$ through a sequence of increasing intermediate times t_1, t_2,..., with corresponding distances $x(t_1)$, $x(t_2)$,..., to the final time $t = 1$ with total distance $x(1)$. We thus assume that $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n \cdots < t_N = 1$ is a sequence of intermediate times with corresponding distances $x(t_n)$ and velocities $f(t_n)$, see Fig. 4.2

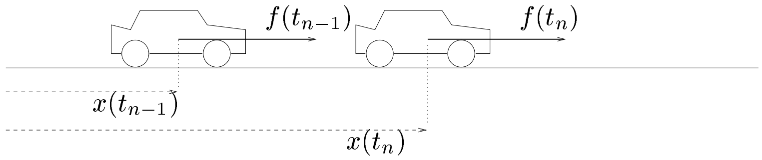

FIGURE 4.2. Distance and velocity at times t_{n-1} and t_n.

For two consecutive times t_{n-1} and t_n, we expect to have

$$x(t_n) \approx x(t_{n-1}) + f(t_{n-1})(t_n - t_{n-1}), \tag{4.3}$$

which says that the distance $x(t_n)$ at time t_n is obtained by adding to the distance $x(t_{n-1})$ at time t_{n-1} the quantity $f(t_{n-1})(t_n - t_{n-1})$, which is the product of the velocity $f(t_{n-1})$ at time t_{n-1} and the *time increment* $t_n - t_{n-1}$. This is because

$$\text{change in distance} \;=\; \text{average velocity} \;\times\; \text{change in time},$$

or traveled distance between time t_{n-1} and t_n equals the (average) velocity multiplied by the time change $t_n - t_{n-1}$. Note that we may equally well connect $x(t_n)$ to $x(t_{n-1})$ by the formula

$$x(t_n) \approx x(t_{n-1}) + f(t_n)(t_n - t_{n-1}), \tag{4.4}$$

corresponding to replacing t_{n-1} by t_n in the $f(t)$-term. We use the approximate equality $\approx$ because we use the velocity $f(t_{n-1})$ or $f(t_n)$, which is not exactly the same as the average velocity over the time interval from t_{n-1} to t_n, but should be close to the average if the time interval is short.

EXAMPLE 4.1. If $x(t) = t^2$, then $x(t_n) - x(t_{n-1}) = t_n^2 - t_{n-1}^2 = (t_n + t_{n-1})(t_n - t_{n-1})$, and (4.3) and (4.4) correspond to approximating the average velocity $(t_n + t_{n-1})$ with $2t_{n-1}$ or $2t_n$, respectively.

The formula (4.3) is at the heart of Calculus! It contains both the derivative of $x(t)$ and the integral of $f(t)$. First, shifting $x(t_{n-1})$ to the left and then dividing by the time increment $t_n - t_{n-1}$, we get

$$\frac{x(t_n) - x(t_{n-1})}{t_n - t_{n-1}} \approx f(t_{n-1}). \tag{4.5}$$

This is a counterpart to (4.2), which indicates how to define the derivative $x'(t_{n-1})$ in order to have the equation $x'(t_{n-1)}) = f(t_{n-1})$ fulfilled:

$$x'(t_{n-1}) \approx \frac{x(t_n) - x(t_{n-1})}{t_n - t_{n-1}}. \tag{4.6}$$

This formula says that the derivative $x'(t_{n-1})$ is approximately equal to the *average velocity*

$$\frac{x(t_n) - x(t_{n-1})}{t_n - t_{n-1}}.$$

over the time interval between t_{n-1} and t_n. Thus, we may expect that the equation $x'(t) = f(t)$ just says that *the derivative $x'(t)$ of the traveled distance $x(t)$ with respect to time t, is equal to the momentary velocity $f(t)$*. The formula (4.6) then says that the velocity $x'(t_{n-1})$ at time t_{n-1}, that is the *momentary velocity* at time t_{n-1}, is approximately equal to the *average velocity* over the time interval (t_{n-1}, t_n). We have now uncovered some of the mystery of the derivative hidden in (4.3).

Next, considering the formula corresponding to to (4.3) for the time instances t_{n-2} and t_{n-1}, obtained by simply replacing n by $n-1$ everywhere in (4.3), we have

$$x(t_{n-1}) \approx x(t_{n-2}) + f(t_{n-2})(t_{n-1} - t_{n-2}), \tag{4.7}$$

and thus together with (4.3),

$$x(t_n) \approx \overbrace{x(t_{n-2}) + f(t_{n-2})(t_{n-1} - t_{n-2})}^{\approx x(t_{n-1})} + f(t_{n-1})(t_n - t_{n-1}). \tag{4.8}$$

Repeating this process, and using that $x(t_0) = x(0) = 0$, we get the formula

$$\begin{aligned} x(t_n) \approx X_n = f(t_0)(t_1 - t_0) + f(t_1)(t_2 - t_1) + \cdots \\ + f(t_{n-2})(t_{n-1} - t_{n-2}) + f(t_{n-1})(t_n - t_{n-1}). \end{aligned} \tag{4.9}$$

Example 4.2. Consider a velocity $f(t) = \frac{t}{1+t}$ increasing with time t from zero for $t = 0$ towards one for large t. What is the travelled distance $x(t_n)$ at time t_n in this case? To get an (approximate) answer we compute the approximation X_n according to (**??**):

$$\begin{aligned} x(t_n) \approx X_n = \frac{t_1}{1+t_1}(t_2 - t_1) + \frac{t_2}{1+t_2}(t_3 - t_2) + \cdots \\ + \frac{t_{n-2}}{1+t_{n-2}}(t_{n-1} - t_{n-2}) + \frac{t_{n-1}}{1+t_{n-1}}(t_n - t_{n-1}). \end{aligned}$$

With a "uniform" time step $k = t_j - t_{j-1}$ for all j, this reduces to

$$\begin{aligned} x(t_n) \approx X_n = \frac{k}{1+k}k + \frac{2k}{1+2k}k + \cdots \\ + \frac{(n-2)k}{1+(n-2)k}k + \frac{(n-1)k}{1+(n-1)k}k. \end{aligned}$$

We compute the sum for $n = 1, 2, .., N$ choosing $k = 0.05$, and plot the resulting values of X_n inFig. 4.3.

We now return to (4.9), and setting $n = N$ we have in particular

$$\begin{aligned} x(1) = x(t_N) \approx f(t_0)(t_1 - t_0) + f(t_1)(t_2 - t_1) + \cdots \\ + f(t_{N-2})(t_{N-1} - t_{N-1}) + f(t_{N-1})(t_N - t_{N-1}), \end{aligned}$$

that is, $x(1)$ is (approximately) the sum of the terms $f(t_{n-1})(t_n - t_{n-1})$ with n ranging from $n = 1$ up to $n = N$. We may write this in more condensed form using the *summation sign* Σ as

$$x(1) \approx \sum_{n=1}^{N} f(t_{n-1})(t_n - t_{n-1}), \tag{4.10}$$

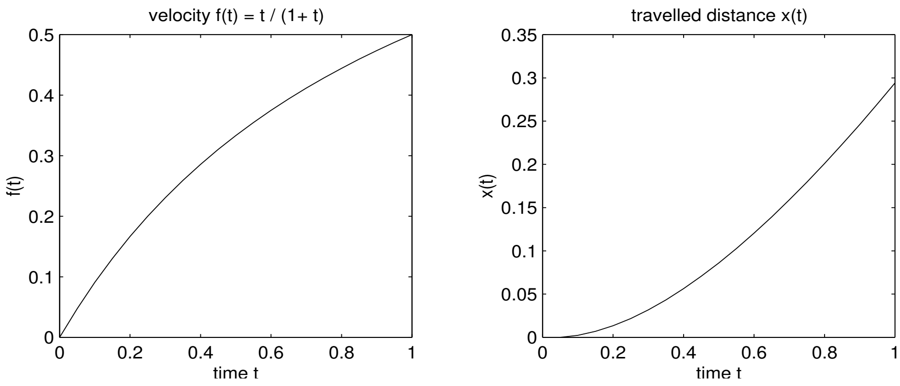

FIGURE 4.3. Travelled distance X_n approximating $x(t)$ for $f(t) = \frac{t}{1+t}$ with time steps $k = 0.05$.

which expresses the total distance $x(1)$ as the sum of all the increments of distance $f(t_{n-1})(t_n - t_{n-1})$ for $n = 1, ..., N$. We can view this formula as a variant of the "telescoping" formula

$$\begin{aligned}
x(1) &= x(t_N)\overbrace{-x(t_{N-1}) + x(t_{N-1})}^{=0}\overbrace{-x(t_{N-2}) + x(t_{N-2})}^{=0}\cdots \\
&\qquad + \overbrace{-x(t_1) + x(t_1)}^{=0}-x(t_0) \\
&= \underbrace{x(t_N) - x(t_{N-1})}_{\approx f(t_{N-1})(t_N - t_{N-1})} + \underbrace{x(t_{N-1}) - x(t_{N-2})}_{\approx f(t_{N-2})(t_{N-1} - t_{N-2})} + x(t_{N-2}) \cdots - x(t_1) \\
&\qquad + \underbrace{x(t_1) - x(t_0)}_{\approx f(t_0)(t_1 - t_0)}
\end{aligned}$$

expressing the total distance $x(1)$ as a sum of all the increments $x(t_n) - x(t_{n-1})$ of distance (assuming $x(0) = 0$), and recalling that

$$x(t_n) - x(t_{n-1}) \approx f(t_{n-1})(t_n - t_{n-1}).$$

In the telescoping formula, each value $x(t_n)$, except $x(t_N) = x(1)$ and $x(t_0) = 0$, occurs twice with different signs.

In the language of Calculus, the formula (4.10) will be written as

$$x(1) = \int_0^1 f(t)\, dt, \tag{4.11}$$

where $\approx$ has been replaced by $=$, the sum $\sum$ has been replaced by the *integral* $\int$, the increments $t_n - t_{n-1}$ by dt, and the sequence of "discrete" time instances t_n, running (or rather "jumping" in small steps) from time 0