59
The Solar System*

> There is talk of a new astrologer who wants to prove that the earth moves and goes around instead of the sky, the sun, the moon, just as if somebody were moving in a carriage or ship might hold that he was sitting still and at rest while the earth and the trees walked and moved. But that is how things are nowadays: when a man wishes to be clever he must needs invent something special, and the way he does it must needs be the best! The fool wants to turn the whole art of astronomy upside-down. However, as Holy Scripture tells us, so did Joshua bid the sun to stand still and not the earth.
> (Sixteenth century reformist M. Luther in his table book *Tischreden*, in response to Copernicus' pamphlet *Commentariolus*, 1514.)

59.1 Introduction

The problem of mathematical modeling of our Solar System including the Sun, the 9 planets Venus, Mercury, Tellus (the Earth), Mars, Jupiter, Saturn, Uranus, Neptune and Pluto together with a large number of moons and asteroids and occasional comets, has been of prime concern for humanity since the dawn of culture. The ultimate challenge concerns mathematical modeling of the Universe consisting of billions of galaxies each one consisting of billions of stars, one of them being our own Sun situated in the outskirts of the Milky Way galaxy.

According to the *geocentric* view presented by Aristotle (384-322 BC) in *The Heavens* and further developed by Ptolemy (87-150 AD) in *The Great*

System dominating the scene over 1800 years, the Earth is the center of the Universe with the Sun, the Moon, the other planets and the stars moving around the Earth in a complex pattern of circles upon circles (so-called epicycles). Copernicus (1473–1543) changed the view in *De Revolutionibus* and placed the Sun in the center in a new *heliocentric* theory, but kept the complex system of epicycles (now enlarged to a very complex system of 80 circles upon circles). Johannes Kepler (1572–1630) discovered, based on the extensive accurate observations made by the Swedish/Danish scientist Tycho Brahe (1546–1601), that the planets move in elliptic orbits with the Sun in one of the foci following *Kepler's laws*, which represented an enormous simplification and scientific rationalization as compared to the system of epicycles.

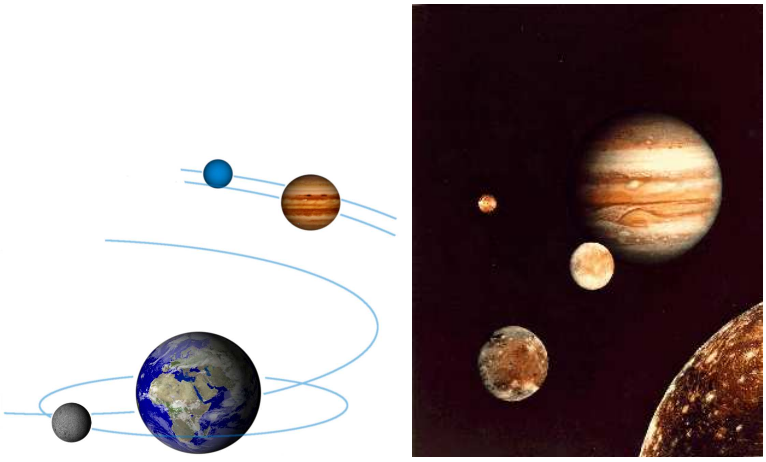

FIGURE 59.1. The Earth with the Moon and some other planets in orbit. Jupiter and the Galilean satellites, Io, Europa, Ganymede, and Callisto

In fact, already Aristarchus (310-230 BC) of Samos understood that the Earth rotates around its axis and thus could explain the (apparent) motion of the stars, but these views were rejected by Aristotle arguing as follows: if the Earth is rotating, how is it that an object thrown upwards falls on the same place? How come this rotation does not generate a very strong wind? No one until Copernicus could question these arguments. Can you?

Newton (1642–1727) then cleaned up the theory by showing that the motion of the planets could be explained from one single hypothesis: the inverse square law of gravitation, see Chapter *Newton's nightmare* below. In particular, Newton derived Kepler's laws for the *two-body problem* with one (small) planet in an elliptic orbit around a (large) sun, see Chapter *Lagrange and the Principle of Least Action.* Leibniz critized Newton for not giving any explanation of the inverse square law, which Leibniz believed could be derived from some basic fact, beyond one of "mutual love" which was quite popular. A sort of explanation was given by Einstein (1879–1955) in his theory of *General Relativity* with gravition arsing as a consequence of space-time being "curved" by the presence of mass. Einstein revolutionized *cosmology*, the theory of the Universe, but relativistic effects only add small corrections to Newton's model for our Solar System based on the inverse square law. Einstein gave no explanation why space-time gets curved by mass, and still today there is no convincing theory of gravitation with its mystical feature of "action at a distance" through some mechanism yet to be discovered. In Chapter *Laplacian models* below we give a derivation of the inverse square law using a mathematical argument presented by Laplace.

Despite the lack of a physical explanation of the inverse square law, Newton's theory gave an enourmous boost to mathematical sciences and a corresponding kick to the egos of scientists: if the human mind was capable of (so easily and definitely) understanding the secrets of the Solar System, then there could be no limits to the possibilities of scientific progress...

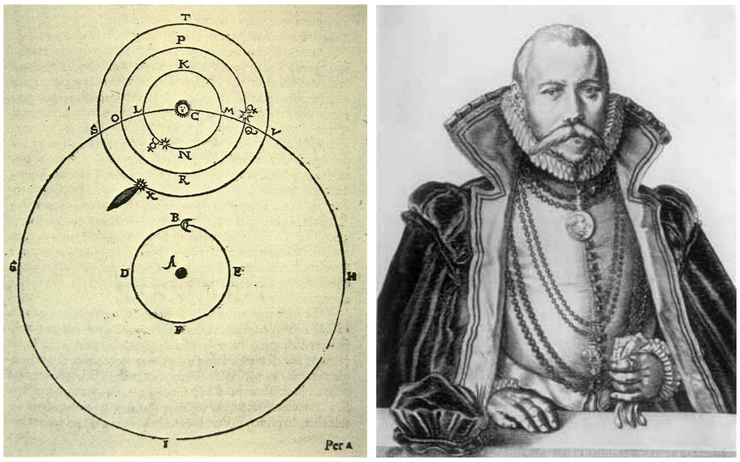

FIGURE 59.2. Tycho Brahe: "I believe that the Sun and the Moon orbit around the Earth but that the other planets orbit around the Sun".

59.2 Newton's Equation

The basis of celestial mechanics is Newton's second law,

$$F = m \cdot a, \tag{59.1}$$

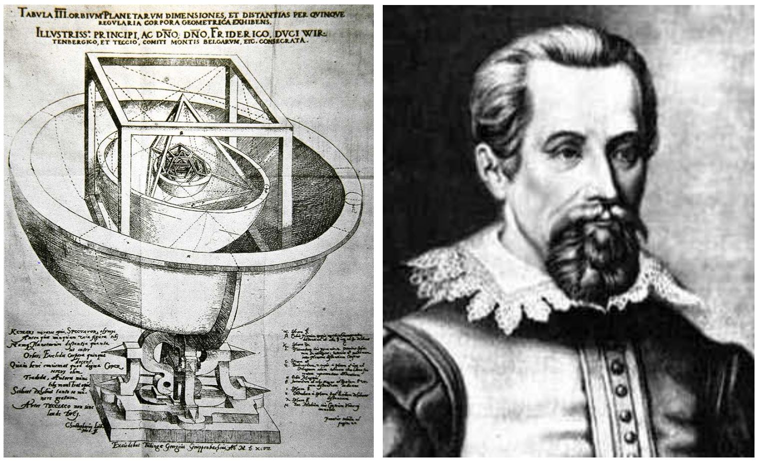

FIGURE 59.3. Johannes Kepler: "I believe that the planets are separated by invisible regular polyhedra: tetrahedron, cube, octahedron, dodekahedron and ikosahedron, and further that the planets including the Earth move in elliptical orbits around the Sun".

expressing that a *force* F results in an acceleration of size a for a body of mass m, together with the expression for the gravitational force given by the inverse square law:

$$F = G\frac{mm_a}{r^2}, \tag{59.2}$$

where $G \approx 6.67 \cdot 10^{-11}\mathrm{Nm}^2/\mathrm{kg}^2$ is the *gravitational constant*, m_a is the mass of the attracting body and r is the distance to the attracting body.

Together (59.1) and (59.2) give a set of differential equations for the evolution of the Solar System. If we know the initial positions and velocities for all bodies in the Solar System, we can solve the system of differential equations, using the same techniques as presented above in Chapter Adaptive IVP-Solvers. We discuss this in detail below in Section 59.4. As a preparation, we rewrite (59.1) and (59.2) in dimensionless form, which will be convenient. The three fundamental units appearing in the equations are those of *space*, *time* and *mass*, which are represented by the variables x (or r), t and m. We now introduce new dimensionless variables, $x' = x/\mathrm{AU}$, $t' = t/\mathrm{year}$ and $m' = m/M$, where 1 AU is the mean distance from the Sun to Earth and M is the mass of the Sun. We can use the chain rule to obtain the dimensionless acceleration, $a' = \frac{d}{dt'}\frac{d}{dt'}x' = \frac{dt}{dt'}\frac{d}{dt}\frac{dt}{dt'}\frac{d}{dt}x/\mathrm{AU} = \frac{\mathrm{year}^2}{\mathrm{AU}}a$. Combining (59.1) and (59.2) using our new dimensionless variables, we then obtain

$$m'M\frac{\mathrm{AU}}{\mathrm{year}^2}a' = G\frac{m'M \cdot m'_aM}{r'^2\mathrm{AU}^2}, \tag{59.3}$$

or

$$a' = G' \frac{m'_a}{r'^2}, \tag{59.4}$$

where the new gravitational constant G' is given by

$$G' = \frac{G \cdot \text{year}^2 M}{\text{AU}^3}. \tag{59.5}$$

We leave it as an exercise to show that with suitable definitions of the units year and AU, the new dimensionless gravitational constant G' is given by

$$G' = 4\pi^2. \tag{59.6}$$

59.3 Einstein's Equation

In general relativity the basic concept is not *force*, as in Newtonian theory, but instead the *curvature* of space-time. Einstein explains the motion of the planets in our Solar System in the following way: the planets move through space-time along straight lines, *geodesics*, which appear as circular (or elliptical) orbits only because space-time is curved by the large mass of the Sun. We shall now try to give an idea of how this works.

The curvature of space-time is given by its *metric*. A metric defines the distance between two nearby points in space-time. In Euclidean geometry that we have studied extensively in this book, the distance between to points $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ is given by the square root of the scalar product $dx \cdot dx$, where dx is the difference $dx = x - y$. With the notation $ds = |x - y|$ we thus have

$$ds = \sqrt{dx \cdot dx} = \left(\sum_{i=1}^{3} dx_i^2 \right)^{1/2}, \tag{59.7}$$

or

$$ds^2 = \sum_{i=1}^{3} dx_i^2. \tag{59.8}$$

In the notation of general relativity, the Euclidean metric is then given by the matrix (tensor)

$$g = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{59.9}$$

as

$$ds^2 = dx^T g \; dx. \tag{59.10}$$

In space-time we include time t as a fourth coordinate and every event in space-time is given by a vector (t, x_1, x_2, x_3). In flat or *Minkowski* space-time in the absence of masses, the curvature is zero and the metric is given by

$$g = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{59.11}$$

which gives

$$ds^2 = -dt^2 + dx_1^2 + dx_2^2 + dx_3^2. \tag{59.12}$$

In the precense of masses, we obtain a different metric which does not even have to be diagonal.

From the metric g one can find the straight lines of space-time, which give the orbits of the planets. The metric itself is determined by the distribution of mass in space-time, and is given by the solution of Einstein's equation,

$$R_{ij} - \frac{1}{2} R g_{ij} = 8\pi T_{ij}, \tag{59.13}$$

where (R_{ij}) is the so-called *Ricci-tensor*, R is the so-called *scalar curvature* and (T_{ij}) is the so-called *stress-energy tensor*. Now (R_{ij}) and R depend on derivatives of the metric $g = (g_{ij})$ so (59.13) is a partial differential equation for the metric g.

The solution for the orbits of the planets obtained from Einstein's equation are a little different than the solution obtained from (59.4) given by Newton. Although the difference is small, it has been verified in observations of the orbit of the planet Mercury which is the planet closest to the Sun. We will not include these "relativistic effects" in the next section where we move on to the computation of the evolution of the Solar System.

59.4 The Solar System as a system of ODEs

To use the techniques developed in Chapter Adaptive IVP-Solvers to compute the evolution of the Solar System, we need to rewrite the second-order system of ODEs given by (59.4) in the standard form $\dot{u} = f$. We start by introducing coordinates $x^i(t) = (x_1^i(t), x_2^i(t), x_3^i(t))$ for all bodies in the Solar System, including the nine planets, then Sun and the Moon. This gives a total of $n = 9 + 2 = 11$ bodies and a total of $3n = 33$ coordinates. To rewrite the equations as the first-order system $\dot{u} = f$ we need to include also the velocities of all bodies, $\dot{x}^i(t) = (\dot{x}_1^i(t), \dot{x}_2^i(t), \dot{x}_3^i(t))$, giving a total of $N = 6n = 66$ coordinates. We collect all these coordinates in the vector $u(t)$ of length N in the following order:

$$\begin{aligned} u(t) = (&x_1^1(t), x_2^1(t), x_3^1(t), \ldots, x_1^n(t), x_2^n(t), x_3^n(t), \\ &\dot{x}_1^1(t), \dot{x}_2^1(t), \dot{x}_3^1(t), \ldots, \dot{x}_1^n(t), \dot{x}_2^n(t), \dot{x}_3^n(t)), \end{aligned} \tag{59.14}$$

so that the first half of the vector $u(t)$ contains the positions of all bodies and the second half contains the corresponding velocities.

To obtain the differential equation for $u(t)$, we take the time-derivative and notice that the derivative of the first half of $u(t)$ is equal to the second half of $u(t)$:

$$\dot{u}_i(t) = u_{3n+i}(t), \quad i = 1, \dots, 3n, \tag{59.15}$$

i.e. for $n = 11$ we have $\dot{u}_1(t) = \dot{x}_1^1(t) = u_{34}(t)$ and so on.

The derivative of the second half of $u(t)$ will contain the second derivatives of the positions, i.e. the accelerations, and these are given by (59.4). Now (59.4) is written as a scalar equation and we have to rewrite it in vector form. For every body in the Solar System, we need to compute the contribution to the total force on the body by summing the contributions from all other bodies. Assuming that we work in dimensionless variables (but writing x instead of x', m_i instead of m_i' and so on for convenience) we then need to compute the sum:

$$\ddot{x}_i(t) = \sum_{j \neq i} \frac{G' m_j}{|x^j - x^i|^2} \frac{x^j - x^i}{|x^j - x^i|}, \tag{59.16}$$

where the unit vector $\frac{x^j - x^i}{|x^j - x^i|}$ gives the direction of the force, see Figure 59.4.

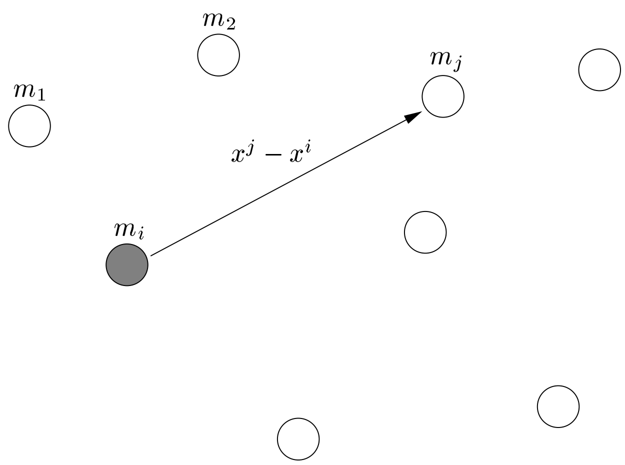

FIGURE 59.4. The total force on body i is the sum of the contributions from all other bodies.

Our final differential equation for the evolution of the Solar System in the form $\dot{u} = f$ is then given by

$$\dot{u}(t) = f(u(t)) = \begin{bmatrix} u_{3n+1}(t) \\ \vdots \\ u_{6n}(t) \\ \sum_{j\neq 1} \frac{G' m_j}{|x^j - x^1|^2} \frac{x_1^j - x_1^1}{|x^j - x^i|} \\ \vdots \\ \sum_{j\neq n} \frac{G' m_j}{|x^j - x^n|^2} \frac{x_3^j - x_3^n}{|x^j - x^n|} \end{bmatrix}, \tag{59.17}$$

where we have kept the notation $x^1 = (x_1^1, x_2^1, x_3^1)$ rather than (u_1, u_2, u_3) and so on in the right-hand side for simplicity. The evolution of our Solar System can now be computed by the standard techniques developed in Chapter Adaptive IVP-Solvers, using the initial data supplied in Table 59.1.

59.5 Predictability and Computability

Two important questions that arise naturally when we study numerical solutions of the evolution of our Solar System, such as the one in Figure 59.5, are the questions of *predictability* and *computability*.

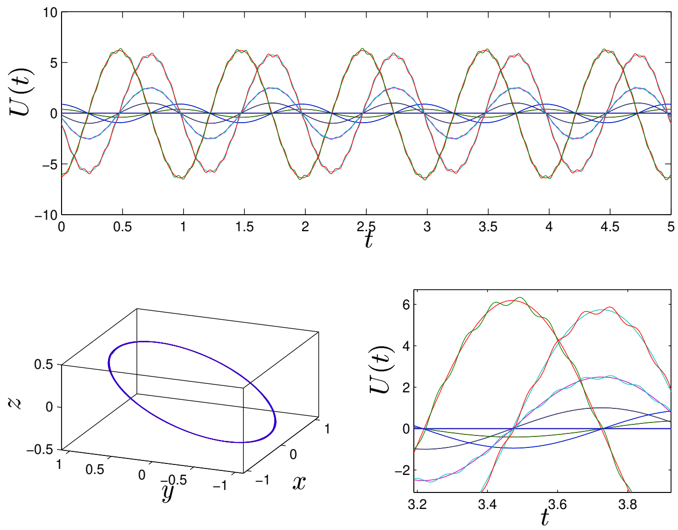

FIGURE 59.5. A numerical computation of the evolution of the Solar System, including Earth, the Sun and the Moon.

	Position	Velocity	Mass
90Mercury	$x^1(0) = \begin{matrix} -0.147853935 \\ -0.400627944 \\ -0.198916163 \end{matrix}$	$\dot{x}^1(0) = \begin{matrix} 7.733816715 \\ -2.014137426 \\ -1.877564183 \end{matrix}$	1.0/6023600
90Venus	$x^2(0) = \begin{matrix} -0.725771746 \\ -0.039677000 \\ 0.027897127 \end{matrix}$	$\dot{x}^2(0) = \begin{matrix} 0.189682646 \\ -6.762413869 \\ -3.054194695 \end{matrix}$	1.0/408523.5
90Earth	$x^3(0) = \begin{matrix} -0.175679599 \\ 0.886201933 \\ 0.384435698 \end{matrix}$	$\dot{x}^3(0) = \begin{matrix} -6.292645274 \\ -1.010423954 \\ -0.438086386 \end{matrix}$	1.0/328900.5
90Mars	$x^4(0) = \begin{matrix} 1.383219717 \\ -0.008134314 \\ -0.041033184 \end{matrix}$	$\dot{x}^4(0) = \begin{matrix} 0.275092348 \\ 5.042903370 \\ 2.305658434 \end{matrix}$	1.0/3098710
90Jupiter	$x^5(0) = \begin{matrix} 3.996313003 \\ 2.731004338 \\ 1.073280866 \end{matrix}$	$\dot{x}^5(0) = \begin{matrix} -1.664796930 \\ 2.146870503 \\ 0.960782651 \end{matrix}$	1.0/1047.355
90Saturn	$x^6(0) = \begin{matrix} 6.401404019 \\ 6.170259699 \\ 2.273032684 \end{matrix}$	$\dot{x}^6(0) = \begin{matrix} -1.565320566 \\ 1.286649577 \\ 0.598747577 \end{matrix}$	1.0/3498.5
90Uranus	$x^7(0) = \begin{matrix} 14.423408013 \\ -12.510136707 \\ -5.683124574 \end{matrix}$	$\dot{x}^7(0) = \begin{matrix} 0.980209400 \\ 0.896663122 \\ 0.378850106 \end{matrix}$	1.0/22869
90Neptune	$x^8(0) = \begin{matrix} 16.803677095 \\ -22.983473914 \\ -9.825609566 \end{matrix}$	$\dot{x}^8(0) = \begin{matrix} 0.944045755 \\ 0.606863295 \\ 0.224889959 \end{matrix}$	1.0/19314
90Pluto	$x^9(0) = \begin{matrix} -9.884656563 \\ -27.981265594 \\ -5.753969974 \end{matrix}$	$\dot{x}^9(0) = \begin{matrix} 1.108139341 \\ -0.414389073 \\ -0.463196118 \end{matrix}$	1.0/150000000
90Sun	$x^{10}(0) = \begin{matrix} -0.007141917 \\ -0.002638933 \\ -0.000919462 \end{matrix}$	$\dot{x}^{10}(0) = \begin{matrix} 0.001962209 \\ -0.002469700 \\ -0.001108260 \end{matrix}$	1
90Moon	$x^{11}(0) = \begin{matrix} -0.177802714 \\ 0.884620944 \\ 0.384016593 \end{matrix}$	$\dot{x}^{11}(0) = \begin{matrix} -6.164023246 \\ -1.164502534 \\ -0.506131880 \end{matrix}$	$1.0/2.674 \cdot 10^7$

TABLE 59.1. Initial data for the Solar System at 00.00 Universal Time (UT1, approximately GMT) January 1 2000 for dimensionless positions and velocities scaled with units 1 AU $= 1.49597870 \cdot 10^{11}$ m (one astronomical unit), 1 year $= 365.24$ days and $M = 1.989 \cdot 10^{30}$ kg (one solar mass).

The predictability of the Solar System is the question of the accuracy of a computation given the accuracy in initial data. If initial data is known with an accuracy of say five digits, and the numerical computation is exact, how long does it take until the solution is no longer accurate even to one digit?

The computability of the Solar System is the question of the accuracy in a numerical solution given exact initial data, i.e. how far we can compute an accurate solution with available resources such as method, computational power and time.

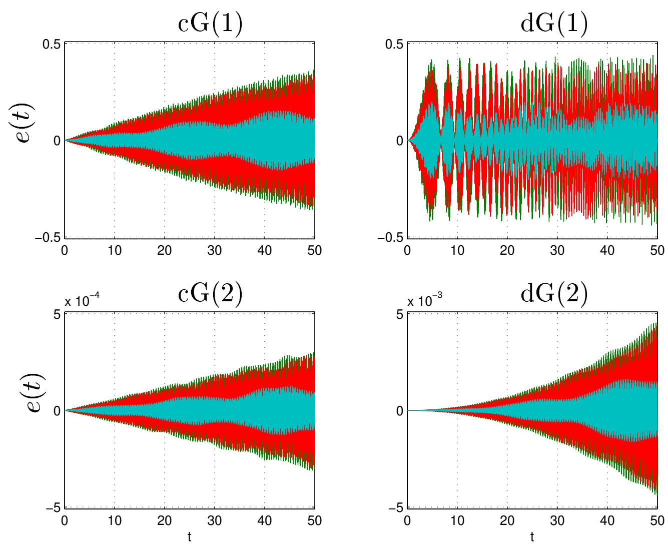

FIGURE 59.6. The growth of the numerical error in simulations of the Solar System using different numerical methods. The two methods on the left conserve energy, which results in linear rather than quadratic error growth.

Both the predictability and the computability are determined by the growth rate of errors. Luckily, the error does not grow exponentially as we saw for the Lorenz system. If we imagine that we displace Earth slightly from its orbit and start a computation, the orbit and velocity of Earth will be slightly different, resulting in an error that grows *linearly* with time. This means that the predictability of the Solar System is quite good, since every extra digit of accuracy in initial data means that the limit of predictability is increased by a factor ten. If now the solution is computed using a numerical method, such as the adaptive cG(1) method, this will result in additional errors. We can think of the error caused by a numerical method as a small perturbation introduced with every new time step. Adding the contributions from all time steps we find that the numerical error grows *quadratically*, see Problem 59.2.

As it turns out however, the error does not grow quadratically but only linearly for the cG(1) method as shown in Figure 59.6. This pleasant surprise is the result of an important property of the cG(1) method: it conserves energy. As a result, the cG(1) method performs better on a long time interval than the higher-order (more accurate) dG(2) method.

59.6 Adaptive time-stepping

If we compute the evolution of the Solar System using the adaptive cG(1) method, we find that the time steps need to be small enough to follow the orbit of the Moon (or Mercury if we do not include the Moon). This is inefficient since the time scales for the other bodies are much larger: the period of the Moon is one month and the period of Pluto is 250 years, and so the time steps for Pluto should be roughly a factor 3,000 larger that the time steps for the Moon. It has been shown recently that the standard methods cG(q), including cG(1), and dG(q) can be extended to individual, *multi-adaptive*, time-stepping for different components. In Figure 59.7 we show a computation made with individual time steps for the different planets. Notice how the error grows quadratically, indicating that the method does not conserve energy. (It is possible to construct also multi-adaptive methods which conserve energy.)

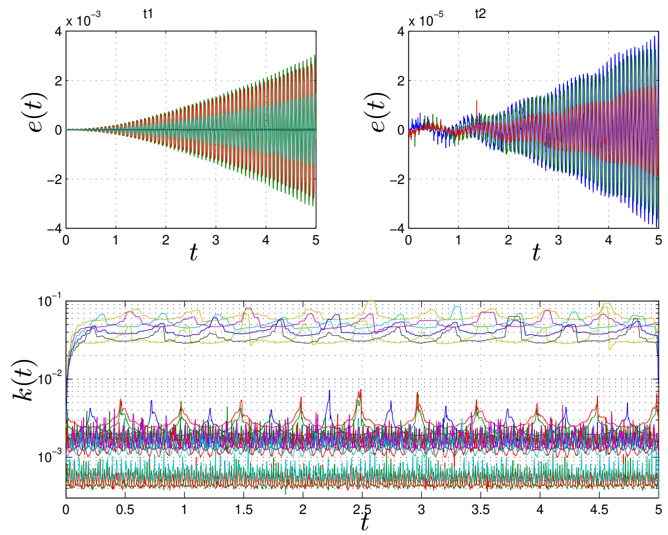

FIGURE 59.7. A computation of the evolution of the Solar System with individual, multi-adaptive, time steps for the different planets.

59.7 Limits of computability and predictability

Using the multiadaptive cG(2) it appears that the limit of computability of the Solar System (with the Moon and the nine planets) using double precision, is of the order 10^6 years. Concerning the predictability of the same system it appears that for every digit beyond 5 in the precision of data we gain a factor of ten in time, so that for example predicting the position of the Moon 1000 years ahead would require about 8 correct digits in e.g. the initial positions and velocities, masses and gravitational constant. We conclude that it appears that normally the precision in data would set the limit for accurate simulations of the evolution of the Solar System, if we use a high order multiadaptive solver.

Chapter 59 Problems

59.1. Prove that with suitable definitions of the units year and AU the gravitational constant is $G' = 4\pi^2$. *Hint:* assume that Earth is in a circular orbit where the centripetal force mv^2/r is balanced by the gravitational force GmM/r^2.

59.2. Motivate the quadratic growth of the numerical error for the Solar System. *Hint:* Assume that an error of size ϵ is added to the velocity of a planet in every time step.

59.3. (*Hard!*) Prove that in general if an error in initial data grows as

$$|e(T)| \leq S(T)|e(0)|,$$

for a specific initial value problem, then the error in a numerical solution of the initial value problem grows as

$$|e(T)| \lessapprox \epsilon \int_0^T S(t)\, dt,$$

assuming that the additional error in every time step is kept below $k_n\epsilon$.

59.4. Prove that the cG(1) method conserves energy for a Hamiltonian system, i.e. prove that for a system given by $\ddot{x} = F = -\nabla_x P(x)$, the total energy

$$E(t) = K(\dot{x}(t)) + P(x(t)),$$

is conserved. Here $P(x)$ is a given potential field, and $K(\dot{x}) = \frac{\dot{x}^2}{2}$ is the kinetic energy. *Hint:* Write as a first-order system for the vector $[u, v] = [x, \dot{x}]$, take $[\dot{v}, \dot{u}]$ as the test function and use the chain rule.

59.5. Investigate numerically the predictability and computability of the Solar System. Can you verify the linear error growth for the cG(1) method?

59.6. Add a comet to your Solar System and try to choose the initial data for the comet so that it (almost) hits the Moon. Can you make the Moon come loose from Earth? Do errors still grow linearly?

> But Aristarchus has brought out a book consisting of certain hypotheses, wherein it appears, as a consequence of the assumptions made, that the universe is many times greater than the 'universe' just mentioned. His hypotheses are that the fixed stars and the sun remain unmoved, that the earth revolves about the sun on the circumference of a circle, the sun lying in the middle of the orbit, and that the sphere of fixed stars, situated about the same centre as the sun, is so great that the circle in which he supposes the earth to revolve bears such a proportion to the distance of the fixed stars as the centre of the sphere bears to its surface. (Archimedes about Aristharcus of Samos)