49
The Crash model*

On October 24, 1929, people began selling their stocks as fast as they could. Sell orders flooded market exchanges. On a normal day, only 750-800 members of the New York Stock Exchange started the Exchange. However, there were 1100 members on the floor for the morning opening. Furthermore, the Exchange directed all employees to be on the floor since there were numerous margin calls and sell orders placed overnight and extra telephone staff was arranged at the members' boxes around the floor. The Dow Jones Industrial Index closed at 299 that day. October 29 was the beginning of the Crash. Within the first few hours the stock market was open, prices fell so far as to wipe out all the gains that had been made in the previous year. The Dow Jones Industrial Index closed at 230. Since the stock market was viewed as the chief indicator of the American economy, public confidence was shattered. Between October 29 and November 13 (when stock prices hit their lowest point) over $30 billion disappeared from the American economy. It took nearly twenty-five years for many stocks to recover. (www.arts.unimelb.edu.au/amu/ucr/student/1997/Yee/1929.htm)

49.1 Introduction

Why did the Wall fall on November 9 1989? Why did the Soviet Union dissolve in January 1992? Why did the Stock market collapse in October 1929 and 1987? Why did Peter and Mary break up last Fall after 35 years of marriage? What caused the September 11 attack? Why does the flow in the river go from orderly laminar to chaotic turbulent at a certain specific point? All the situations behind these questions share a common feature: Nothing particularly dramatic preceded the sudden transition from stable to unstable, and in each case the rapid and dramatic change away from normality came as big surprise to almost everyone.

We now describe a simple mathematical model that shows the same behavior: the solution stays almost constant for a long time and then quite suddenly the solution explodes.

We consider the following initial value problem for a system of two ordinary differential equations: find $u(t) = (u_1(t), u_2(t))$ such that

$$\begin{cases} \dot{u}_1 + \epsilon u_1 - \lambda u_2 u_1 = \epsilon & t > 0, \\ \dot{u}_2 + 2\epsilon u_2 - \epsilon u_1 u_2 = 0 & t > 0, \\ u_1(0) = 1, u_2(0) = \kappa\epsilon, \end{cases} \tag{49.1}$$

where ϵ is a small positive constant of size say 10^{-2} or smaller and λ and κ are positive parameters of moderate size ≈ 1. If $\kappa = 0$, then the solution $u(t) = (1, 0)$ is constant in time, which we view as the *base solution.* In general, for $\kappa > 0$, we think of $u_1(t)$ as a primary part of solution with initial value $u_1(0) = 1$, and $u_2(t)$ as a small secondary part with an initial value $u_2(0) = \kappa\epsilon$ that is small because ϵ is small. Both components $u_1(t)$ and $u_2(t)$ will correspond to physical quantities that are non-negative and $u_1(0) = 1$ and $u_2(0) = \kappa\epsilon \geq 0$.

49.2 The simplified growth model

The system (49.1) models an interaction between a primary quantity $u_1(t)$ and a secondary quantity $u_2(t)$ through the terms $-\lambda u_1 u_2$ and $-\epsilon u_2 u_1$. If we keep just these terms, we get a simplified system of the form

$$\begin{cases} \dot{w}_1(t) = \lambda w_1(t) w_2(t) & t > 0, \\ \dot{w}_2(t) = \epsilon w_2(t) w_1(t) & t > 0, \\ w_1(0) = 1, \quad w_2(0) = \kappa\epsilon. \end{cases} \tag{49.2}$$

We see that the coupling terms are *growth terms* in the sense that both the equation $\dot{w}_1(t) = \lambda w_1(t) w_2(t)$ and $\dot{w}_2(t) = \epsilon w_2(t) w_1(t)$ say that $\dot{w}_1(t)$ and $\dot{w}_2(t)$ are positive if $w_1(t) w_2(t) > 0$. In fact, the system (49.1) always blow up for $\kappa > 0$ because the two components propel each other to infinity as t increases in the sense that the right hand sides get bigger with $w_1(t) w_2(t)$ and this increases the growth rates $\dot{w}_1(t)$ and $\dot{w}_2(t)$, which in turn makes $w_1 w_2(t)$ even bigger, and so on towards blow up, see Fig. 49.1.

We can study the blow up in (49.2) analytically assuming for simplicity that $\lambda = \kappa = 1$. In this case, it turns out that the two components $w_1(t)$ and

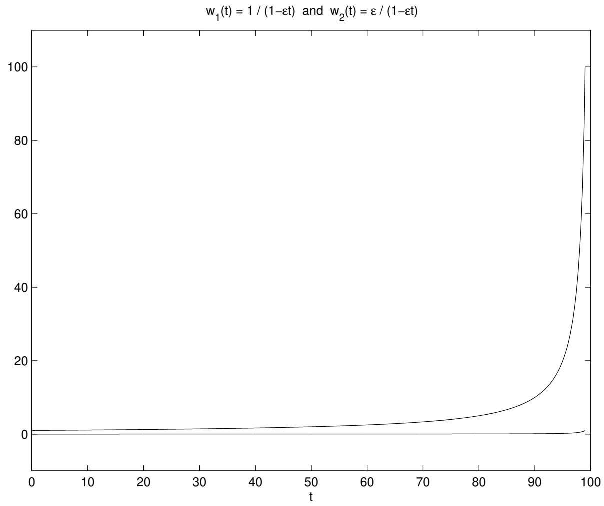

FIGURE 49.1. Solution of simpified growth model

$w_2(t)$ for all t are coupled by the relation $w_2(t) = \epsilon w_1(t)$, that is $w_2(t)$ is always the same multiple of $w_1(t)$. We check this statement by first verifying that $w_2(0) = \epsilon w_1(0)$ and then by dividing the two equations to see that $\dot{w}_2(t)/\dot{w}_1(t) = \epsilon$. So, $\dot{w}_2(t) = \epsilon \dot{w}_1(t)$, that is $w_2(t) - w_2(0) = \epsilon w_1(t) - \epsilon w_2(0)$, and we get the desired conclusion $w_2(t) = \epsilon w_1(t)$ for $t > 0$. Inserting this relation into the first equation of (49.2), we get

$$\dot{w}_1(t) = \epsilon w_1^2(t) \qquad \text{for } t > 0,$$

which can be written as

$$-\frac{d}{dt}\frac{1}{w_1(t)} = \epsilon \qquad \text{for } t > 0.$$

Recalling the initial condition $w_1(0) = 1$, we get

$$-\frac{1}{w_1(t)} = \epsilon t - 1 \qquad \text{for } t \geq 0,$$

which gives the following solution formula in the case $\lambda = \kappa = 1$:

$$w_1(t) = \frac{1}{1 - \epsilon t}, \quad w_2(t) = \frac{\epsilon}{1 - \epsilon t} \qquad \text{for } t \geq 0. \tag{49.3}$$

This formula shows that the solution tends to infinity as t increases towards $1/\epsilon$, that is, the solution explodes at $t = 1/\epsilon$. We notice that the time of blow up is $1/\epsilon$, and that the *time scale* before the solution starts to increase noticeably, is of size $\frac{1}{2\epsilon}$, which is a long time since ϵ is small. Thus, the solution changes very slowly for a long time and then eventually blows up quite a bit more rapidly, see Fig. 49.1.

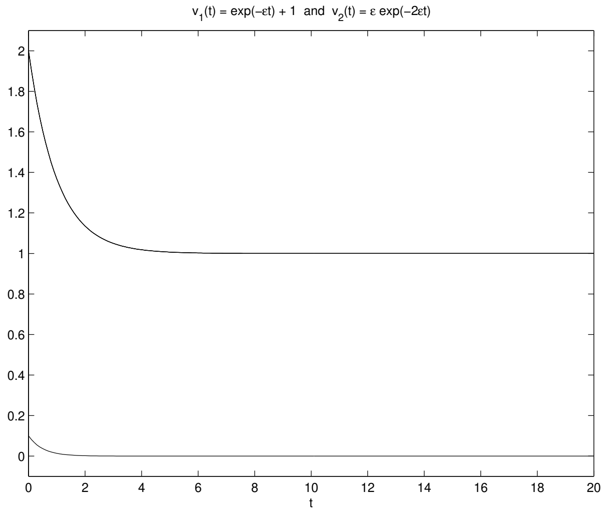

FIGURE 49.2. Solution of simpified decay model

49.3 The simplified decay model

On the other hand, if we forget about the growth terms, we get another simplified system:

$$\begin{cases} \dot{v}_1 + \epsilon v_1 = \epsilon \qquad t > 0, & t > 0, \\ \dot{v}_2 + 2\epsilon v_2 = 0 & t > 0, \\ v_1(0) = 1 + \delta, \quad v_2(0) = \kappa\epsilon, \end{cases} \tag{49.4}$$

where we have also introduced a small perturbation δ in $v_1(0)$. Here the two terms ϵv_1 and $2\epsilon v_2$ are so called *dissipative* terms that cause the solution $v(t)$ to return to the base solution $(1, 0)$ regardless of the perturbation, see Fig. 49.2. This is clear in the equation $\dot{v}_2 + 2\epsilon v_2 = 0$ with solution $v_2(t) = v_2(0)\exp(-2\epsilon t)$, which decays to zero as t increases. Rewriting the equation $\dot{v}_1 + \epsilon v_1 = \epsilon$ as $\dot{V}_1 + \epsilon V_1 = 0$, setting $V_1 = v_1 - 1 = \exp(-\epsilon t)$, we find that $v_1(t) = \delta\exp(-\epsilon t) + 1$, and thus $v_1(t)$ approaches 1 as t increases. We summarize: the solution $(v_1(t), v_2(t))$ of (49.6) satisfies

$$v_1(t) = \delta\exp(-\epsilon t) + 1 \to 1, \quad v_2(t) = \kappa\epsilon\exp(-2\epsilon t) \to 0, \quad \text{as } t \to \infty.$$

We say that (49.6) is a *stable* system because the solution always returns from $(1+\delta, \kappa\epsilon)$ to the base solution $(1, 0)$ independently of the perturbation $(\delta, \kappa\epsilon)$ of $(v_1(0), v_2(0))$.

We note that the time scale is again of size $1/\epsilon$, because of the presence of the factors $\exp(-\epsilon t)$ and $\exp(-2\epsilon t)$.

49.4 The full model

We can now sum up: The real system (49.1) is a combination of the unstable system (49.2) that includes only the growth terms only and whose the solution always blows up, and the stable system (49.6) that excludes the growth terms. We shall see that depending on the size of $\lambda\kappa$ the unstable or stable feature will take over. In Fig. 49.3 and Fig. 49.4, we show different solutions for different values of the parameters λ and κ with different initial values $u(0) = (u_1(0), u_2(0)) = (1, \kappa\epsilon)$. We see that if $\lambda\kappa$ is sufficiently large, then the solution $u(t)$ eventually blows up after a time of size $1/\epsilon$, while if $\lambda\kappa$ is sufficiently small, then the solution $u(t)$ returns to the base solution $(1, 0)$ as t tends to infinity.

Thus, there seems to be a *threshold value* for $\lambda\kappa$ above which the initially disturbed solution eventually blows up and below which the initially disturbed solution returns to the base solution. We can view κ as a measure of the size if the initial disturbance, because $u_2(0) = \kappa\epsilon$. Further, we can view the factor λ as a quantitative measure of the *coupling* between the growth components $u_2(t)$ and $u_1(t)$ through the growth term $\lambda u_1 u_2$ in the evolution equation for u_1.

Our main conclusion is that if the initial disturbance times the coupling is sufficiently large, then the system will blow up. Blow up thus requires both the initial disturbance and the coupling to be sufficiently large. A large initial disturbance will not cause blow up unless there some coupling. A strong coupling will not cause blow up unless there is an initial disturbance.

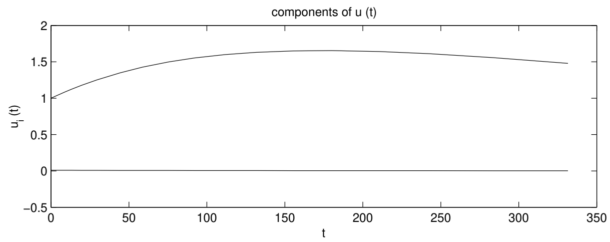

FIGURE 49.3. Return to the base solution if $\lambda\kappa$ is small enough

We now investigate the qualitative behavior of (49.1) in a little more detail. We see that $\dot{u_1}(0)/u_1(0) = \lambda\kappa\epsilon$, while $\dot{u_2}(0)/u_2(0) = -\epsilon$, which shows that initially $u_1(t)$ grows and $u_2(t)$ decays at relative rates of size ϵ. Now, $u_1(t)$ will continue to grow as long as $\lambda u_2(t) > \epsilon$, and further $u_2(t)$ will start to grow as soon as $u_1(t) > 2$. Thus, if $u_1(t)$ manages to become larger than 2, before $u_2(t)$ has decayed below ϵ/λ, then both components will propel each other to cause a blow up to infinity. This happens if $\lambda\kappa$ is above a certain threshold.

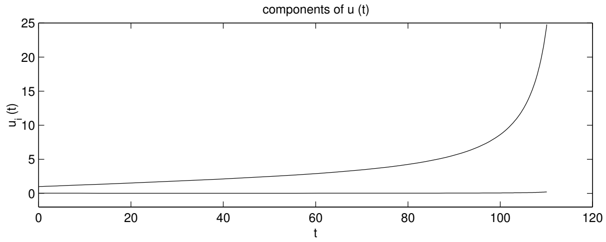

FIGURE 49.4. Blow up if $\lambda\kappa$ is large enough

We notice that the time scale for significant changes in both u_1 and u_2 is of size ϵ^{-1}, because the growth rates are of size ϵ. This conforms with the experience from the simplified models. The scenario is thus that the primary part $u_1(t)$ grows slowly starting from 1 at a rate of size ϵ and the secondary part $u_2(t)$ decays slowly at a rate of size ϵ^2, over a time of size $1/\epsilon$. If $\lambda\kappa$ is above a certain threshold, then $u_1(t)$ reaches the value 2, at which point $u_2(t)$ starts to grow and eventually blow up follows on a somewhat shorter time scale. If $u_1(t)$ does not reach the value 2 in time, then $(u_1(t), u_2(t))$ returns to the base solution $(1, 0)$ as t increases.

We hope the presented scenario is quite easy to grasp intuitively, and conforms with every-day experiences of quit sudden blow-up, as a result of an accumulation of small events over a long period.

We can give the Crash model very many interpretations in real life, such as

- stock market (u_1 stock prize of big company, u_2 stock prize of small innovative company),
- chemical reaction (u_1 main reactant, u_2 catalyst),
- marriage crisis (u_1 main discontent, u_2 small irritation factor),
- spread of infection (u_1 infected people, u_2 amount of germs),
- symbiosis (u_1 main organism, u_2 small parasite),
- population model (u_1 rabbits, u_2 vitalizing carrots),

and many others.

In particular, the model describes an essential aspect of the process of transition from laminar to turbulent flow in for example a pipe. In this case u_1 represents a flow component in the direction of the pipe and u_2 represents a small perturbation of the flow in the transversal direction. The time to explosion corresponds to the time it takes for the flow to go turbulent starting as laminar flow at the inlet. In the famous experiment of Reynolds from 1888, ink is injected at the inlet of a transparent pipe and the observer can follow the streamline traced by the ink, which forms a straight line in the laminar part and then successively becomes more and more wavy until it breaks down to completely turbulent flow at some distance from the inlet. The distance to breakdown varies with the flow speed and viscosity and perturbations resulting from e.g. roughness of the surface of the pipe or a heavy-weight truck passing by at some distance from the experimental set-up.

Chapter 49 Problems

49.1. Develop the indicated applications of the Crash model.

49.2. Solve the full system (49.1) numerically for various values of λ and κ and try to pin down the threshold value of $\lambda\kappa$.

49.3. Develop a *Theory of Capitalism* based on (49.1) as a simple model of the economy in a society, with u_1 representing the value of a basic resource like land, and u_2 some venture capital related to the exploitation of new technology, with $(1,0)$ a base solution without the new technology, and with the coeffcient λ of the u_1u_2 term in the first equation representing the positive interplay between base and new technology, and the terms ϵu_i representing stabilizing effects of taxes for example. Show that the possible pay-off $u_1(t) - u_1(0)$ of a small investment $u_2(0) = \kappa\epsilon$ may be large, and that an exploding economy may result if $\lambda\kappa$ is large enough. Show that no growth is possible if $\lambda = 0$. Draw some conclusions from the model coupled to for example the role of the interest rate for controlling the economy.

49.4. Interprete (49.1) as a simple model of a stock market with two stocks, and discuss scenarios of overheating. Extend to a model for the world stock market, and predict the next crash.

49.5. Consider the linear model

$$\begin{aligned}
\dot{\varphi}_1 + \epsilon\varphi_1 - \lambda\varphi_2 &= 0 \qquad t > 0,\\
\dot{\varphi}_2 + \epsilon\varphi_2 &= 0 \qquad t > 0,\\
\varphi_1(0) = 0, \quad \varphi_2(0) &= \kappa\epsilon,
\end{aligned} \tag{49.5}$$

which is obtained from (49.1) by setting $\varphi_1 = u_1 - 1$ and $\varphi_2 = u_2$ and replacing $u_1\varphi_2$ by φ_2 assuming u_1 is close to 1. Show that the solution of (49.5) is given by

$$\varphi_2(t) = \kappa\epsilon\exp(-\epsilon t), \quad \varphi_1(t) = \lambda\kappa\epsilon t\exp(-\epsilon t).$$

Conclude that

$$\frac{\varphi_1(\frac{1}{\epsilon})}{\varphi_2(0)} = \lambda\frac{\exp(-1)}{\epsilon},$$

and make an interpretation of this result.

49.6. Expand the Crash model (49.1) to

$$\begin{aligned} \dot{u}_1 + \epsilon u_1 - \lambda u_1 u_2 + \mu_1 u_2^2 &= \epsilon \qquad t > 0, \\ \dot{u}_2 + 2\epsilon u_2 - \epsilon u_2 u_1 + \mu_2 u_1^2 &= 0 \qquad t > 0, \\ u_1(0) = 1, \quad u_2(0) &= \kappa\epsilon, \end{aligned} \tag{49.6}$$

with decay terms $\mu_1 u_2^2$ and $\mu_2 u_1^2$, where μ_1 and μ_2 are positive coefficients. (a) Study the stabilizing effect of such terms numerically. (b) Seek to find values of μ_1 and μ_2, so that the corresponding solution starting close to $(1, 0)$ shows an intermittent behaviour with repeated periods of blow up followed by a decay back to a neigborhood of $(1, 0)$. (c) Try to find values of μ_1 and μ_2 so that multiplication of the first equation with a positive multiple of u_1 and the second by u_2, leads to bounds on $|\epsilon u_1(t)|^2$ and $|u_2(t)|^2$ in terms of initial data. Hint: Try for example $\mu_1 \approx 1/\epsilon$, and $\mu_2 \approx \epsilon^2$.

49.7. Study the initial value problem $\dot{u} = f(u)$ for $t > 0$, $u(0) = 0$, where $f(u) = \lambda u - u^3$, with different values of $\lambda \in \mathbb{R}$. Relate the time-behaviour of $u(t)$ to the set of solutions $\bar{u}$ of $f(u) = 0$, that is, $\bar{u} = 0$ if $\lambda \leq 0$, and $\bar{u} = 0$ or $\bar{u} = \pm\sqrt{\lambda}$ if $\lambda > 0$. Study the linearized models $\dot{\varphi} - \lambda\varphi + 3\bar{u}^2\varphi = 0$ for the different $\bar{u}$. Study the behaviour of the solution assuming $\lambda(t) = t - 1$.

49.8. Study the model

$$\begin{aligned} \dot{w}_1 + w_1 w_2 + \epsilon w_1 &= 0, \qquad t > 0, \\ \dot{w}_2 - \epsilon w_1^2 + \epsilon w_2 &= -\gamma\epsilon, \qquad t > 0, \end{aligned} \tag{49.7}$$

with given initial data $w(0)$, where γ is a parameter and $\epsilon > 0$. This problem admits the stationary "trivial branch" solution $\bar{w} = (0, -\gamma)$ for all γ. If $\gamma > \epsilon$, then also $\bar{w} = (\pm\sqrt{\gamma - \epsilon}, -\epsilon)$ is a stationary solution. Study the evolution of the solution for different values of γ. Study the corresponding linearized problem, linearized at $\bar{w}$.